# NUMERICAL-ANALYTICAL AND SIMULATION MODELING OF A QUEUING SYSTEM WITH HYPER-ERLANG DISTRIBUTION OF THE INPUT FLOW

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**This paper presents the results of a study of a queueing system (QS) described by a hyper-Erlangian input flow and exponential service time. The hyper-Erlangian distribution is known to be a special case of phase distributions, which have become a mathematical tool for analyzing telecom traffic in future-generation networks, characterized by high variability and exhibiting self-similarity. To construct a numerical analytical model of the average waiting time of requests in a queue, a spectral method was used to solve the Lindley integral equation, finding the zeros and poles of the spectral decomposition as a ratio of rational functions. To build a simulation model, the logical switch method in the GPSS World discrete-event simulation system was used. This paper presents numerical-analytical and simulation models for the average queue wait time for requests in a system with a second-order hyper-Erlang distribution for the input flow. Constructing the simulation model required developing a pseudo-random sequence generator for a second-order hyper-Erlang distribution, which is not available in the GPSS World function library. The adequacy of the developed generator's performance was confirmed by statistical tests. To assess the accuracy of the resulting solution, computational experiments were conducted for small, medium, and high values of the time interval variation coefficients and system load. The adequacy of the results is confirmed by a comparison of the simulation data using the presented numerical-analytical and simulation models and computational experiments in Mathcad.**

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## Introduction

A fundamental principle of general systems theory states that the average queue wait time is nonlinearly dependent on the coefficients of variation of arrival and service time intervals [1]. Consequently, the wider the range of variability of these coefficients of variation, ensured by the distribution law, the better suited the queuing system (QS) is for traffic modeling. These requirements are satisfied by the hyperexponential and hyper-Erlang distributions, which are special cases of phase distributions that are widely used to model highly variable traffic.

Modern telecommunications traffic varies greatly over time, making hyperexponential and hyper-Erlang distributions vital for its accurate modeling. In this article, in addition to constructing a numerical-analytical model of queuing using the hyper-Erlang distribution law, we also consider the need for such models in the GPSS WORLD system, proposing a new solution based on the logical switch mechanism. The universal discrete-event simulation system GPSS WORLD is used to model various production systems that can be represented as queuing systems. Its library contains numerous programs, including software random number generators based on the standard U(0,1) uniform distribution.

Due to the difficulty of developing generators for probability mixture distribution laws, this system does not include generators for phase distributions, including hyper-Erlang and hyper-exponential distributions as probability mixture distributions. The cited literature and many other scientific publications confirm the lack of openly available results similar to those presented by the authors [2-7]. This fact complicates the modeling of general-purpose G/G/1 and G/G/m queueing systems, which best describe modern telecom traffic.

The results presented in this article are intended to partially address the aforementioned shortcomings of the GPSS WORLD modeling system.

It is known, for example [2], that the hyper-Erlang distribution can be constructed as a probability mixture of normalized Erlang distributions with weight coefficients $p$ and 1-$p$. For example, the second-order hyper-Erlangian density function has the form

$$f(t) = 4p\lambda_1^2 te^{-2\lambda_1 t} + 4(1-p)\lambda_2^2 te^{-2\lambda_2 t}, \tag{1}$$

and provides the coefficient of variation $c \in (1/\sqrt{2}, \infty)$.

From the definition of the two-parameter gamma distribution with density function $f(t) = \begin{cases} \dfrac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)}, & t \ge 0; \\ 0, & t < 0, \end{cases}$, it follows that for a positive integer value of $k$ ($\alpha$=$k$), the distribution with density function

$$f(t) = \lambda^2 te^{-\lambda t} \tag{2}$$

represents the Erlang distribution of order $k$. The numerical characteristics of distribution (2) depend on its order $k$, which creates certain problems when approximating flat distributions by the Erlang law. However, such problems do not arise when using the normalized Erlang distribution. The library generator of the gamma distribution GAMMA allows generating random sequences with the distribution (2), so we use (2) in the density function of the hyper-Erlang distribution as a probability mixture

$$f(t) = p\lambda_1^2 te^{-\lambda_1 t} + (1-p)\lambda_2^2 te^{-\lambda_2 t}. \tag{3}$$

In [2], it is noted that at high coefficients of variation, (1) and (3) can describe heavy-tailed distributions. Furthermore, [2] provides considerable detail about the hyper-Erlang distribution, establishing that the density function (1) can be viewed as an alternative representation of the second-order hyper-Erlang density function (3). This article is devoted to the construction of a simulation model of a QS with input distribution (3) in the GPSS WORLD system with a comparison with a numerical-analytical model. To this end, this paper presents a new method for generating second-order hyper-Erlang distributions in the GPSS WORLD modeling system, filling a gap in available tools for modeling variable telecommunication traffic.

As no such generator currently exists, this work provides a novel method, using a logical switches mechanism. In the future, this approach will allow us to demonstrate high-dimensional examples instead of simple examples. The wide range of coefficients of variation limits the possibilities of simulating data transmission systems for various purposes, because these distributions are underestimated. Similar work with hyperexponential input distribution is presented in [10]. It should be noted that the hyper-Erlang distribution has a wider range for the coefficient of variation than the hyper-exponential distribution.

The article also uses methods and techniques for approximating distribution laws using the initial moments of distributions [8-14]. Relatively new results in the field of simulation modeling are presented in [15-16].

## 1. Problem Statement

We consider a queuing system $HE_2/M/1$, which is described by the distribution density functions of arrival intervals $HE_2$:

$$a(t) = p\lambda_1^2 te^{-\lambda_1 t} + (1-p)\lambda_2^2 te^{-\lambda_2 t} \tag{4}$$

and service time M:

$$b(t) = \mu e^{-\mu t}. \tag{5}$$

The article sets the task of constructing a QS simulation model with input distribution (3) in the GPSS WORLD system and comparing its results with a numerical-analytical model. To this end, this paper also aims to construct a method for generating a second-order hyper-Erlang distribution in the GPSS WORLD modeling system to fill the gap in available tools for modeling highly variable telecom traffic.

## 2. Theoretical solution for the $HE_2/M/1$ system

Theoretical analysis for the $HE_2/M/1$ system with distribution $HE_2$ of the form (1) was carried out in [8]. As noted above, distributions (1) and (3) differ in numerical characteristics. Therefore, the analytical model of the system under consideration shown below differs from the model considered in [8] and is also presented for the first time. The spectral decomposition method [1] was used to obtain a theoretical solution of the problem for the $HE_2/M/1$ system. Let us first write down the Laplace transforms for distributions (4) and (5):

$$A^*(s)=p(\frac{\lambda_1}{\lambda_1+s})^2+(1-p)(\frac{\lambda_2}{\lambda_2+s})^2,\ B^*(s)=\frac{\mu}{s+\mu}.$$

For the considered system, the spectral expansion takes the form

$$\frac{\alpha(s)}{\beta(s)}=\left[p\left(\frac{\lambda_1}{\lambda_1-s}\right)^2+(1-p)\left(\frac{\lambda_2}{\lambda_2-s}\right)^2\right]\cdot\left(\frac{\mu}{\mu+s}\right)-1.$$

To continue the expansion and reduction of expressions, we introduce intermediate parameters

$$a_0=\lambda_1^2\lambda_2^2,\ a_1=2\lambda_1\lambda_2[p\lambda_1+(1-p)\lambda_2],\ a_2=p\lambda_1^2+(1-p)\lambda_2^2.$$

The spectral expansion will take the form

$$\frac{\alpha(s)}{\beta(s)}=\frac{\mu(a_0-a_1s+a_2s^2)-(\lambda_1-s)^2(\lambda_2-s)^2(\mu+s)}{(\lambda_1-s)^2(\lambda_2-s)^2(\mu+s)}=$$

$$=\frac{-s(s^4+c_3s^3+c_2s^2+c_1s+c_0)}{(\lambda_1-s)^2(\lambda_2-s)^2(\mu+s)}=\frac{-s(s+s_1)(s-s_2)(s-s_3)(s-s_4)}{(\lambda_1-s)^2(\lambda_2-s)^2(\mu+s)}.$$

Finally

$$\frac{\alpha(s)}{\beta(s)}=\frac{-s(s+s_1)(s-s_2)(s-s_3)(s-s_4)}{(\lambda_1-s)^2(\lambda_2-s)^2(\mu+s)}. \tag{6}$$

To find the zeros and poles of expansion (6), we select a polynomial of the fourth degree from the numerator

$$s^4+c_3s^3+c_2s^2+c_1s+c_0 \tag{7}$$

with coefficients collected using symbolic operations Mathcad

$$c_0=a_1\mu+a_0-2\mu\lambda_1\lambda_2(\lambda_1+\lambda_2),$$

$$c_1=\mu(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2-a_2)-2\lambda_1\lambda_2(\lambda_1+\lambda_2),$$

$$c_2=\lambda_1^2+\lambda_2^2+4\lambda_1\lambda_2-2\mu(\lambda_1+\lambda_2),\ c_3=\mu-2(\lambda_1+\lambda_2).$$

Polynomial (7) has one real negative root $-s_1$. The next three roots in this case represent one real positive and two complex conjugates with a positive real part. Next, we determine the components of the expansion based on the type of roots of the numerator of the expansion

$$\alpha(s)=s(s+s_1)/(\mu+s),\ \beta(s)=-\frac{(2\lambda_1-s)^2(2\lambda_2-s)^2}{(s-s_2)(s-s_3)(s-s\ \ )}.$$

Using the found components of the spectral decomposition, we determine the Laplace transform for the waiting time density function $W^*(s)=\dfrac{s_1(s+\mu)}{\mu(s+s_1)}$, and through it, the average waiting time in the queue

$$\overline{W}=1/s_1-1/\mu. \tag{8}$$

To determine the value of the root s1 of the polynomial (7), it is necessary to express the values of the unknown parameters $\lambda_1$, $\lambda_2$, $p$ of the distribution (4) and $\mu$ of the distribution (5) through its initial moments:

$$\overline{\tau}_\lambda=\frac{2p}{\lambda_1}+\frac{2(1-p)}{\lambda_2},\ \overline{\tau_\lambda^2}=\frac{6p}{\lambda_1^2}+\frac{6(1-p)}{\lambda_2^2}, \tag{9}$$

$$\overline{\tau}_\mu=\frac{1}{\mu},\ \overline{\tau_\mu^2}=\frac{2}{\mu^2}.$$

Equations (10), together with the squared coefficient of variation $c_\lambda^2=\sigma_\lambda^2/(\overline{\tau}_\lambda)^2=[\overline{\tau_\lambda^2}-(\overline{\tau}_\lambda)^2]/(\overline{\tau}_\lambda)^2$, make it possible to find all three distribution parameters (4). Let's write these values

$$\lambda_1=4p/\overline{\tau}_\lambda,\ \lambda_2=4(1-p)/\overline{\tau}_\lambda,$$

$$p=1/2\pm\sqrt{1/4-3/[8(1+c_\lambda^2)]}.$$

(10)

In this case, any of the two numbers can be taken as the value of the parameter $p$. From the condition of non-negativity of the expression under the square root, it follows that $c_\lambda\geq 1/\sqrt{2}$ [8-9].

A similar solution for the $HE_2/M/1$ system is presented in the author's paper [8]; for the case where the $HE_2$ distribution is formulated as a probability mixture of normalized Erlang (1) distributions. As established previously, the normalized distribution (1) and the usual distribution (3) have different initial moments. Consequently, many intermediate results (7) differ from those given in this article. Nonetheless, the primary result for the average waiting time remains the same.

Below in Figure 1 is an example of a calculation using the proposed QS model $HE_2/M/1$ for the case of a load factor $\rho=\overline{\tau}_\mu/\overline{\tau}_\lambda=0.9$, a coefficient of variation $c_\lambda=2$ and a unit service time $\overline{\tau}_\mu=1$.

$$\tau\lambda:=\frac{10}{9}\quad c\lambda:=2\quad \tau\mu:=1\quad c\mu:=1\quad \mu:=\frac{1}{\tau\mu}=1$$

$$p:=\frac{1}{2}+\sqrt{\frac{[2\cdot(1+c\lambda^2)-3]}{8\cdot(1+c\lambda^2)}}=0.9183\quad \lambda1:=4\cdot\frac{p}{\tau\lambda}=3.306\quad \lambda2:=\frac{4\cdot(1-p)}{\tau\lambda}=0.294$$

$$a0:=\lambda1^2\cdot\lambda2^2\quad a1:=2\cdot\lambda1\cdot\lambda2\cdot[p\cdot\lambda1+(1-p)\cdot\lambda2]\quad a2:=p\cdot\lambda1^2+(1-p)\cdot\lambda2^2$$

$$C3:=\mu-2\cdot(\lambda1+\lambda2)=-6.2\quad C2:=-2\cdot\mu\cdot(\lambda1+\lambda2)+4\cdot\lambda1\cdot\lambda2+(\lambda1^2+\lambda2^2)=7.704$$

$$C1:=\mu\cdot(\lambda1^2+4\cdot\lambda1\cdot\lambda2+\lambda2^2-a2)-2\cdot\lambda1\cdot\lambda2\cdot(\lambda1+\lambda2)=-2.138$$

$$C0:=a0+a1\cdot\mu-2\cdot\mu\cdot\lambda1\cdot\lambda2\cdot(\lambda1+\lambda2)=-0.105$$

Given

$$S^4+C3\cdot S^3+C2\cdot S^2+C1\cdot S+C0=0$$

$$\text{Find}(S)\rightarrow(1.13E+000\quad 4.64E+000\quad -4.239E-002\quad 4.722E-001)$$

$$S1:=0.04239365392405731241 5\quad S2:=5.0877761192984934606$$

$$S3:=0.0336792844885409462 76\quad S4:=0.0081293249893227337 75$$

$$W:=\frac{1}{S1}-\frac{1}{\mu}=22.58844\quad Nq:=\frac{W}{\tau\lambda}=20.33$$

**Fig. 1.** Calculation example on Mathcad

Figure 1 presents an example of a numerical calculation of the model, which can be used to evaluate the numerical calculation algorithm.

In conclusion of the construction of the numerical-analytical model, we present Table 1, which compares the results for our system in the case of a relatively small variation coefficient $c\lambda=2$ with the data of known systems for different values of the load ρ.

Table 1

A simple database for comparing systems

| | Average waiting time | | |
|---|---|---|---|
| ρ | $HE_2/M/1$ | M/M/1 | $E_2/M/1$ |
| 0,1 | 0,08 | 0,11 | 0,03 |
| 0,5 | 2,00 | 1,00 | 0,62 |
| 0,9 | 22,59 | 9,00 | 6,59 |

### 3. Construction of a simulation model for the system under consideration

An analytical representation of the hyper-Erlang distribution (4) suggests an initial approach to modeling using the GPSS World TEST operator. This is a conditional jump operator depending on whether a given condition is met. In such a model, transactions will be directed with probability *p* to the first phase of the hyper-Erlang law and with probability 1-*p* to the second phase, as shown in the model text below. The simulation model code is presented in Appendix 1. Here the line HYPER1 FVARIABLE means the command to call the generator of the second-order Erlang distribution as a special case of the Gamma distribution and assign the calculated value to the arithmetic variable HYPER1.

The results of the simulation with the initial data from the Matcad calculations (Fig. 1) are shown in Fig. 2. To maintain continuity between the numerical-analytical and simulation models, we use the same initial data for the system during simulation.

```
LABEL   LOC BLOCK TYPE  ENTRY  COUNT  CURRENT  COUNT  RETRY
          1  TEST                0             0      0
          2  GENERATE       917939             0      0
          3  TRANSFER       917939             0      0
MET_1     4  GENERATE        82079             0      0
MET_2     5  QUEUE         1000018            17      0
          6  SEIZE         1000001             1      0
          7  DEPART        1000000             0      0
          8  ADVANCE       1000000             0      0
          9  RELEASE       1000000             0      0
         10  TERMINATE     1000000             0      0

FACILITY  ENTRIES  UTIL.  AVE.TIME  AVAIL.  OWNER   PEND  INTER  RETRY  DELAY
CHAN      1000001  0.901  1.001     1       999992  0     0      0      17

QUEUE  MAX  CONT.  ENTRY    ENTRY(0)  AVE.CONT.  AVE.TIME  AVE.(-0)  RETRY
QCHAN  69   18     1000018  128658    6.012      6.678     7.665     0
```

**Fig. 2.** Results of the initial simulation model run

The analysis of the model run results (Fig. 2) shows that approximately 918 thousand transactions are sent to the first phase, while 82 thousand transactions to the second phase. This distribution corresponds precisely to the specified probabilities of 0.918 and 0.082. As shown in Figure 2, the load factor is 0.9 and the average service time is one, which is completely consistent with the given initial data in Figure 1. At the same time, the main result, namely the average waiting time, equal to 6.678, does not coincide with the results of the analytical model (Fig. 1), which is 22.588 units of model time.

Thus, we came to the need to consider the differences between the analytical and simulation approaches. The issue arises from the structure of the simulation model. In its current implementation, incoming requests are not delayed in the Erlang phases of the hyper-Erlang distribution law for a given phase operation time. Consequently, the queueing delays produced by the simulation are characteristic of a simpler $E_2/M/1$ system, not the intended $HE_2/M/1$ system. We will consider this approach to constructing a simulation model as an instructive (unsuccessful) example.

To construct an adequate simulation model, it is necessary to clarify the main difference between the theoretical hyper-Erlang distribution (4) and the discrete-event functioning of the two-phase distribution. In a theoretical distribution, a request entering the system for servicing instantly goes through an "either or" scheme with probabilities *p* and 1-*p* – the first phase or the second. In discrete event modeling, before a requirement moves into a particular phase, that phase (state) must be defined and opened for a certain time using logical switches. Therefore, the simulation circuit must include logical switches with two states: on and off. Such a simulation circuit is shown in Figure 3.

In the discrete-event model, the two-phase functioning of the hyper-Erlang process (Fig. 3) occurs as follows. With probability *p*, the generated transaction is sent to the first phase of the hyper-Erlang process after it is determined and transferred to the on state and remains there for a time of $2/\lambda_1$, and with probability 1-*p*, it is sent to the second phase and remains there for $2/\lambda_2$ units of model time. The circuit includes logical switches with two states: on, off.

The quantities $2/\lambda_1$ and $2/\lambda_2$ here represent the first initial moments of the second-order Erlang distribution for the first and second phases.

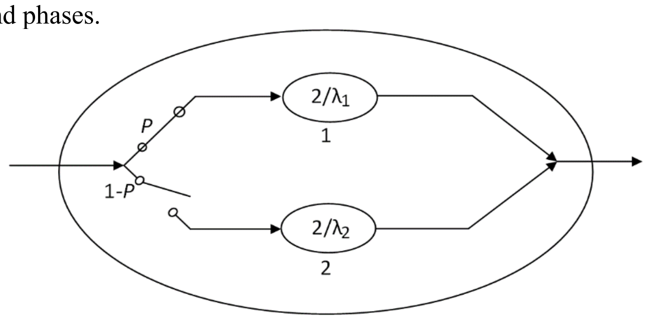


**Fig. 3.** Formation of a Two-Phase Hyper-Erlang Distribution. The phases are switched on alternately depending on the probability *p* and work for an arbitrary time with an average value indicated on the diagram

The two states are controlled by the LOGIC operator, where the A parameter serves as the key's identifier. The operating mode is specified by the letter X: the active state is set using the S (set) value, and deactivation is performed using the R (reset) value. Depending on the given probability variable *p*, the program code provides for the sequential activation of Switch1 and Switch2 objects, which act as logical keys.

The use of switches alone does not complete the model. It is necessary to ensure that transactions proceed through phases only when the switch is active. For this purpose, a GATE condition checker (in let-in mode) is used, which regulates the flow depending on the state of object A. Modeling requires specifying the duration of active switch operation. For example, the expression T_1#3.1 defines the operating time, where the average phase interval is adjusted by an empirical factor of 3.1. This factor is selected empirically to minimize deviations in the QS characteristics from actual data. The GPSS World program code is given in Appendix 2.

Next, we will establish the initial data for the simulation model, which must coincide with the data of the numerical-analytical model. To do this, we take a loading factor equal to 0.9, a variation coefficient for the input flow of 2.0, and distribution parameters (4) determined by expressions (10)

$$p = \frac{1}{2} - \sqrt{\frac{2(1+c_\lambda^2)-3}{8(1+c_\lambda^2)}} = 0{,}082,\; 1-p = 0.918,\; \lambda_1 = 0.295,\; \lambda_2 = 3.304.$$

Let's establish a unit model service time. All these initial data can be found in the program code in Appendix 2. Figure 4 shows the simulation results.

Notations used in the simulation model.

P_1 is a probability $p$ from the theoretical model, P_2 is a probability 1-$p$. T_1 and T_2 here are half the values of the averages $2/\lambda_1$ and $2/\lambda_2$ for the first and second phases of the hyper-Erlang distribution.

Other notations are related to features of the GPSS WORLD modeling language, such as block labels and library function input parameters for generators.

| FACILITY | ENTRIES | UTIL. | AVE.TIME | AVAIL. | OWNER | PEND | INTER | RETRY | DELAY |
|---|---|---|---|---|---|---|---|---|---|
| CHAN | 1000001 | 0.898 | 1.000 | 1 | 2008099 | 0 | 0 | 0 | 1 |

| QUEUE | MAX | CONT. | ENTRY | ENTRY(0) | AVE.CONT. | AVE.TIME | AVE.(-0) | RETRY |
|---|---|---|---|---|---|---|---|---|
| QCHAN | 175 | 2 | 1000002 | 52010 | 20.300 | 22.623 | 23.864 | 0 |

**Fig. 4.** The simulation results for QS $HE_2/M/1$ with a control constant of 3,1. Here the designations of the standard GPSS WORLD report are saved

The simulation results shown in Figure 4 confirm that the given input data and the simulation results match. The key performance indicators of the system, such as the average queue waiting time of 22.623-time units, is close to the numerical result of 22.59, and the average queue length of 20.30 is also almost the same as the numerical result of 20.33 in Figure 1. Thus, the experimental selection of the durations of active operation of switches T1 and T2 allows us to construct an adequate simulation model. Thus, the control constant in the simulation model is optimized, which ensures the best output characteristics of the system. In the future, this may help in constructing general simulation models for PH/PH/1 systems with phase distributions.

We now analyze the simulation model's behavior with the control constant set below and above 3.1. First, for values less than 3.1 (Figure. 5 – control constant is equal to 3.0), second, for values above than 3. 1 (Fig. 6 – control constant is equal to 3.2).

| FACILITY | ENTRIES | UTIL. | AVE.TIME | AVAIL. | OWNER | PEND | INTER | RETRY | DELAY |
|---|---|---|---|---|---|---|---|---|---|
| CHAN | 1000001 | 0.898 | 1.000 | 1 | 2006416 | 0 | 0 | 0 | 0 |

| QUEUE | MAX | CONT. | ENTRY | ENTRY(0) | AVE.CONT. | AVE.TIME | AVE.(-0) | RETRY |
|---|---|---|---|---|---|---|---|---|
| QCHAN | 191 | 1 | 1000001 | 51574 | 19.749 | 21.990 | 23.186 | 0 |

**Fig. 5.** The simulation results for QS $HE_2/M/1$ with a control constant of 3,0

| FACILITY | ENTRIES | UTIL. | AVE.TIME | AVAIL. | OWNER | PEND | INTER | RETRY | DELAY |
|---|---|---|---|---|---|---|---|---|---|
| CHAN | 1000001 | 0.899 | 1.000 | 1 | 2005184 | 0 | 0 | 0 | 13 |

| QUEUE | MAX | CONT. | ENTRY | ENTRY(0) | AVE.CONT. | AVE.TIME | AVE.(-0) | RETRY |
|---|---|---|---|---|---|---|---|---|
| QCHAN | 238 | 14 | 1000014 | 50137 | 20.944 | 23.307 | 24.537 | 0 |

**Fig. 6.** The simulation results for QS $HE_2/M/1$ with a control constant of 3,2

Thus, the model demonstrates the expected relationship where an increase in service parameters leads to a decrease in waiting time and queue length, and vice versa. A practical limitation of the proposed approach, however, is its dependence on knowing the exact or approximate values of the system's output characteristics in advance. Future research could focus on developing methods to bypass and eliminate this drawback.

## 4. Quality check of submitted pseudo-random number Generator and verification of the simulation model

The problem of checking the quality of the proposed pseudo-random variable generator for the hyper-Erlang distribution law is that this distribution law is completely absent from the libraries of known programs for processing statistical data for the purpose of selecting the corresponding distribution law. Consequently, exporting the results of the proposed generator to any known software products is impossible Therefore, to test the quality of the generated sequences, we will have to re-solve the main problem of recognition and selection of the distribution law using the null hypothesis $H_0$.

To do this, we must add to the program code given in Appendix 2 a set of command operators for generating random intervals of incoming requests and processing them in the form of tables and histograms. Let's list them: InterArr TABLE X$Arr,1,1,20 (Building a table of receipt interval values for a statistical series and a histogram); INITIAL X$Time,0 (Setting the initial value of "Time" to zero); Fun VARIABLE (AC1-X$Time) (Determination of the function for the values of interarrivals); SAVEVALUE Arr,V$Fun (Enter in the cell "Arr" the number determined by the function "Fun"); SAVEVALUE Time,AC1 (Enter in the cell "Time" the current value of the absolute model time); TABULATE InterArr (Tabulation of interval values between arrivals); MARK 2 (Enter in cell "2" the value of the current absolute model time). We restrict ourselves to a small number of tests 500. It should be noted that such number of tests will reflect the transient process of the system functioning. The START 500 statement will generate 505 transactions then (Fig. 7).

This could be avoided if there was a possibility to export the generation results into well-known statistical data processing programs.

It is known that varying the width of the histogram digit (Fig. 8) allows one to obtain various Pearson statistics. We combine discharges with low relative frequencies so that at least 5 values of a random variable fall into each interval, as shown in Table 2 [18-20].

| QUEUE | MAX | CONT. | ENTRY | ENTRY(0) | AVE.CONT. | AVE.TIME | AVE.(-0) | RETRY |
|---|---|---|---|---|---|---|---|---|
| QCHAN | 50 | 5 | 505 | 15 | 17.614 | 18.612 | 19.181 | 0 |

| TABLE | MEAN | STD.DEV. | RANGE | RETRY | FREQUENCY | CUM.% |
|---|---|---|---|---|---|---|
| INTERARR | 1.172 | 2.159 | | 0 | | |
| | | | - 1.000 | | 391 | 77.43 |
| | | | 1.000 - 2.000 | | 69 | 91.09 |
| | | | 2.000 - 3.000 | | 10 | 93.07 |
| | | | 3.000 - 4.000 | | 5 | 94.06 |
| | | | 4.000 - 5.000 | | 4 | 94.85 |
| | | | 5.000 - 6.000 | | 8 | 96.44 |
| | | | 6.000 - 7.000 | | 6 | 97.62 |
| | | | 7.000 - 8.000 | | 3 | 98.22 |
| | | | 8.000 - 9.000 | | 2 | 98.61 |
| | | | 9.000 - 10.000 | | 3 | 99.21 |
| | | | 10.000 - 11.000 | | 1 | 99.41 |
| | | | 11.000 - 12.000 | | 1 | 99.60 |
| | | | 12.000 - 13.000 | | 1 | 99.80 |
| | | | 13.000 - 14.000 | | 0 | 99.80 |
| | | | 14.000 - 15.000 | | 0 | 99.80 |
| | | | 15.000 - 16.000 | | 1 | 100.00 |

**Fig. 7.** Statistical distribution of arrival intervals for QS $HE_2/M/1$. Part of the standard GPSS World report is shown here. With a small number of tests, we have a case of a transient process

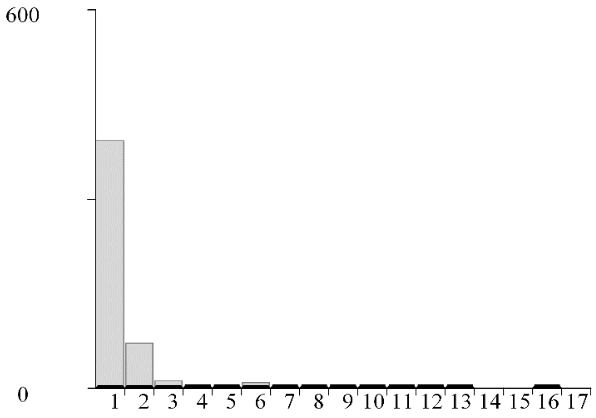


**Fig. 8**. $HE_2$ distribution histogram. The histogram is built using standard GPSS WORLD tools: Window → Simulation Window → Table Window

Here *I* is a digits of the histogram, mi is the number of hits in the *i*-th digit, $p_i$* are the relative frequencies of hits, $p_i$ are the probabilities of hits in the given digit $p_i = \int_{t_{i-1}}^{t_i} [p\lambda_1^2 te^{-\lambda_1 t} + (1-p)\lambda_2^2 te^{-\lambda_2 t}]dt$ . The value of the Pearson statistics is determined using the expression $\chi^2 = \sum_{i=1}^{K} \frac{(m_i - p_i \cdot N)^2}{p_i \cdot N}$ .

Table 2

Corrected statistical series and calculation of statistics $\chi^2$

| *I* | $i$ | $p_i$* | $p_i$ | 505 $p_i$ | $\chi^2$ |
|---|---|---|---|---|---|
| 0-1 | 391 | 0,7743 | 0,7763 | 392,03 | 0,003 |
| 1-2 | 69 | 0,1366 | 0,1423 | 71,86 | 0,114 |
| 2-3 | 10 | 0,0198 | 0,0173 | 8,74 | 0,183 |
| 3-5 | 9 | 0,0178 | 0,0177 | 8,94 | 0,000 |
| 5-6 | 8 | 0,0158 | 0,0077 | 3,89 | 4,347 |
| 6-7 | 6 | 0,0119 | 0,0068 | 3,43 | 1,917 |
| 7-9 | 5 | 0,0099 | 0,0108 | 5,45 | 0,038 |
| 9-∞ | 7 | 0,0139 | 0,0211 | 10,66 | 1,254 |
| | ∑=505 | ∑=1,0 | ∑=1,0 | ∑=505 | ∑=7,86 |

The obtained test value $\chi^2$=7.86 is compared with the table values with the number of degrees of freedom *r*=4 to determine the level of significance. A quantile (significance level) of 0.05 chi squared distribution with four degrees of freedom gives the value $\chi^2$(0.05; 4)=9.49. The calculated value $\chi^2$=7.86 is less than 9.49, therefore, the hypothesis of the hyper-Erlang distribution law for this sample is not rejected. Here, the null hypothesis $H_0$ is the fact that the values of this sample are subject to the hyper-Erlang distribution. The alternative hypothesis $H_1$ – the hyper-Erlang distribution law is rejected. Thus, the reliability of the simulation model is confirmed by comparison with the results of analytical modeling obtained by the spectral solution method presented in [8-10].

To increase the reliability of the results, we apply the Kolmogorov-Smirnov statistical criterion. For this purpose, the empirical distribution function $F_n(t)$ for the request inter-arrival intervals was constructed using the $p_i$* from Table 2. The results of these calculations are presented in Table 3.

Table 3

**Data for the Kolmogorov-Smirnov test**

| Points | $F_n(t)$ | $F(t)$ | $F_n(t)$ - $F(t)$ |
|---|---|---|---|
| 0 | 0,0 | 0,0 | 0,0 |
| 1 | 0,7743 | 0,7763 | -0,0020 |
| 2 | 0,9109 | 0,9186 | -0,0077 |
| 3 | 0,9307 | 0,9359 | -0,0052 |
| 5 | 0,9485 | 0,9536 | -0,0151 |
| 6 | 0,9643 | 0,9613 | 0,0030 |
| 7 | 0,9762 | 0,9681 | 0,0081 |
| 9 | 0,9861 | 0,9789 | 0,0072 |
| ∞ | 1,0 | 1,0 | 0,0 |

As shown in Table 3, the test statistic is $d$=max $|F_n(t) - F(t)|$=0,015 and the Kolmogorov function parameter is $\lambda = d\sqrt{n}$=0,337. Considering the values of the Kolmogorov function $P$(0,3)=1,000 and $P$(0,4)=0,997 after interpolation, we get the probability $P$=0,998. Thus, the hypothesis of the hyperexponential distribution law is supported.

## 5. Results and Discussion

The paper presents the developed algorithm and simulation model for modeling the functioning of a QS with a second-order composite distribution formed from Erlang distributions with weights. It was noted above that this distribution law provides a wide range of the coefficient of variation. When developing the simulation model, a complex mechanism of logical switching of the phases of the distribution law was used, as well as the selection of a control constant, which made it possible to build an adequate simulation model for the passage of transactions within the model. At the same time, the control constant allows you to reduce the time of the transient process during simulation. Checking the quality of the presented generator on small samples shows their satisfactory functioning (Table 2).

## 6. Prospects for further research

Despite the relatively high accuracy of the constructed numerical-analytical and simulation models of queuing for a system with a hyper-Erlang distribution of the input flow, the authors see the need for further development of this direction for systems with phase distributions.

The presented models are limited by the possibility of using only two phases of the incoming process and two moments of distribution of the input flow, and this may be insufficient for modeling modern telecom traffic.

The transition to phase distributions is associated with the complexity of constructing a simulation model with the specifics of using logical switches for the phase structure of receipt and service processes.

## Conclusion

This work addresses a gap in theory, as no prior studies on this topic were found in either Russian or English sources, nor are hyper-Erlang distribution generators available in the GPSS World library. The results presented by the authors make it possible to study queueing systems based on complex distribution laws. The approach involves comparing simulation results with theoretical results obtained by applying the spectral decomposition method to the Lindley integral equation.

In the article, Section 3, including all subsections and Section 4, contain new results that are not available in the Russian-language or English-language segments of scientific literature. This concerns numerical-analytical modeling and simulation modeling of systems with hyper-Erlang input distribution. Therefore, these sections present new results in computer modeling. The presented hyper-Erlang distribution generator based on logical switches of Erlang distribution phases operation has no analogues.

The authors confirm that the data supporting the findings of this study are available within the article [8-10] its supplementary materials.

## Appendix 1

```
10 HYPER1 FVARIABLE (GAMMA(1,0,0.605,2))
20 HYPER2 FVARIABLE (GAMMA(1,0,6.78,2))
30 TEST L (RN1),0.918,MET_1
40 GENERATE V$HYPER1
50 TRANSFER ,MET_2
60 MET_1 GENERATE V$HYPER2
70 MET_2 QUEUE QCHAN
80 SEIZE CHAN
90 DEPART QCHAN
100 ADVANCE (Exponential (1,0,1.0))
110 RELEASE CHAN
120 TERMINATE 1
130 START 1000000
```

## Appendix 2

```
10 P_1 EQU 0.082
20 P_2 EQU (1-P_1)
30 T_1 EQU 3.39
40 T_2 EQU 0.302
50 GENERATE ,,,1
60 Switch TRANSFER P_1,Met_2,Met_1
70 Met_1 TRANSFER ,Sw_1
80 Met_2 TRANSFER ,Sw_2
90 Sw_1 LOGIC S Switch1
100 ADVANCE (T_1#3.1)
110 LOGIC R Switch1
120 TRANSFER ,Switch
130 Sw_2 LOGIC S Switch2
140 ADVANCE (T_2#3.1)
150 LOGIC R Switch2
160 TRANSFER ,Switch
170 GENERATE (GAMMA(11,0,T_1,2))
180 GATE LS Switch1,Met_10
190 TRANSFER ,Met_20
200 GENERATE (GAMMA(21,0,T_2,2))
210 GATE LS Switch2,Met_10
220 Met_20 QUEUE QCHAN
230 SEIZE CHAN
240 DEPART QCHAN
250 ADVANCE (Exponential(31,0,1.0))
260 RELEASE CHAN
270 TERMINATE 1
280 Met_10 TERMINATE
290 START 1000000
```

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# ЧИСЛЕННО-АНАЛИТИЧЕСКОЕ И ИМИТАЦИОННОЕ МОДЕЛИРОВАНИЕ СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ С ГИПЕРЭРЛАНГОВСКИМ РАСПРЕДЕЛЕНИЕМ ВХОДНОГО ПОТОКА

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**Аннотация**
В работе представлены полученные результаты исследования системы массового обслуживания (СМО), описываемой гиперэрланговским входным потоком и экспоненциальным временем обслуживания. Известно, что гиперэрланговское распределение является частным случаем фазовых распределений, которые стали математическим аппаратом для анализа телетрафика сетей будущего поколения, характеризующегося высокой вариативностью и обладающего эффектом самоподобия. Для построения численно-аналитической модели среднего времени ожидания требований в очереди использован спектральный метод для решения интегрального уравнения Линдли с нахождением нулей и полюсов спектрального разложения в виде отношения рациональных функций. Для построения имитационной модели использован метод логических переключателей в дискретно-событийной системе моделирования GPSS World. В работе представлены численно-аналитическая и имитационная модели для среднего времени ожидания требований в очереди для системы с гиперэрланговским распределением второго порядка для входного потока. Для построения имитационной модели потребовалась разработка генератора псевдослучайных последовательностей для гиперэрланговского распределения второго порядка, которого нет в библиотеке функций GPSS World. Адекватность функционирования разработанного генератора подтверждена статистическим тестами. Для оценки точности полученного решения проведены вычислительные эксперименты для малых, средних и высоких значений коэффициентов вариаций временных интервалов и загрузки системы. Адекватность результатов подтверждается сопоставлением данных моделирования с помощью представленных численно-аналитической и имитационной моделей и вычислительными экспериментами в Mathcad.

***Ключевые слова:*** *Система моделирования GPSS WORLD, системы массового обслуживания, среднее время ожидания в очереди, средняя длина очереди, уравнение Линдли, спектральный метод.*

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