OPTIMISATION METHOD FOR CHAINMAIL-TYPE BUSLAEV NETS ON A TORUS UNDER TOPOLOGY VARIATIONS

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The paper is devoted to the study and optimization of Buslaev contour networks of the chainmail type on a torus, which represent discrete dynamical systems with a complex topological structure. These networks explicitly consider conflicts in the shared nodes of contours, making them particularly suitable for modelling traffic flows on complex urban road networks where intersections and overlapping traffic streams play a critical role in determining the overall system capacity and stability. The study presents a comprehensive combinatorial analysis of the network topology, including the determination of the number of diagonals, the distribution of contours across diagonals, and the derivation of the spectrum of average particle velocities. Analytical formulas are developed for estimating the number of allowed states and admissible velocity configurations, taking into account the spatial and topological constraints inherent in the system. A simulation-based investigation is performed for chainmail networks, demonstrating clear correlations between directional movement configurations and the resulting average velocities of particles. A topology optimization method is proposed, enabling the identification of optimal directional patterns that maximize average system velocity while mitigating local conflicts. The obtained results confirm theoretical predictions regarding the minimum achievable velocities and reveal structurally stable optimal configurations that significantly enhance system throughput. Practical recommendations are formulated for applying the developed models to optimize traffic flows in real-world urban street and road networks, offering potential for improving transport efficiency under complex topological and operational constraints.

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1 Introduction

Traffic flow modeling is a fundamental research area in transportation engineering and applied mathematics, aimed at understanding, predicting, and optimizing the dynamics of object movement within complex network structures. Historically, the development of traffic models began with simplified representations, such as the Biham-Middleton-Levine (BML) model [1], which utilized cellular automata to simulate vehicle movement on a square grid. The BML model, despite its conceptual simplicity and clarity, effectively demonstrated phenomena of self-organization and collapse in transportation systems [2]. However, its limitations became apparent when attempting to apply it to more complex and large-scale real-world transportation networks, which are characterized by a high degree of interconnectedness and dynamic variability [3].

Modern transportation systems represent multi-dimensional, heterogeneous networks where the interaction between moving objects and infrastructure creates non-linear effects and complex behavioral patterns [4]. This necessitated the development of more sophisticated models capable of accounting for these complexities. In this context, significant contributions were made by the works of Professor A.P. Buslaev and his research group, who proposed innovative approaches to describing flows in complex networks [5]. Their studies revealed the limitations of traditional agent-based simulation models, which, while highly detailed, suffer from high dimensionality, sensitivity to input data inaccuracies, and unpredictable error growth with an increasing number of agents. These drawbacks make analytical study of such models practically impossible [6].

In response to these challenges, there arose a need for intermediate-level models that, possessing a limited number of parameters, allow for both simulation and analytical study [7]. Such an approach enables a balance between detail and computational efficiency, which is critically important for analyzing large-scale systems. Initially, similar methodologies were applied to analyze movement on circular and linear sections, closely intertwining with cellular automata theory, dynamical systems, and simulation modeling methods [8]. The development of these concepts led to the emergence of contour networks, which, unlike classical models such as BML, offer a more flexible and powerful basis for modeling and analyzing traffic in large and complex networks. In [9] it is proved that the BML model is a special case of Buslaev networks. The constraints imposed on the network structure and movement rules in contour networks facilitate simplified analytical study, allowing for more accurate prediction of traffic behavior and evaluation of system efficiency under various scenarios.

This article is dedicated to the further development of contour network theory, specifically, the study of "chainmail" type contour networks on a torus. The focus is on the mathematical description of these systems, the analysis of their combinatorial properties, and the investigation of the influence of contour movement direction on the average velocity of movement across the chainmail. The goal of this work is to develop a method for optimizing a chainmail type contour network, which can be used to enhance the throughput and modeling efficiency of traffic flows in real-world road networks.

2 Buslaev contour networks

In the works of V.V. Kozlov, A.P. Buslaev et al. introduced a class of dynamical systems called Buslaev contour networks [5] and developed approaches to the study of this class of system. These approaches involve the use of the apparatus of dynamical systems theory, functional analysis, cellular automata, Markov chains with discrete or continuous time, synchronous and asynchronous processes with prohibitions, mass service theory, number-theoretic and algebraic methods. In addition to applications in modelling traffic flows, the results of the study of systems of this class can be used in the analysis of info-communication systems, in materials science, etc.

A contour network is a system of closed contours connected in a network structure. Connectivity between the contours is provided by common connection points – nodes. Both discrete and continuous variants of such networks are considered in the framework of research.

In [10] the notion of a chain as a discrete in time and set of states dynamic system, the carrier of which is a system of $1 \times n$ contours (Fig. 1), was introduced. For a contour C_i the neighboring contours are C_{i-1}, C_{i+1}, $i = 1, \ldots, n$. The chain can be implemented in closed and open versions.

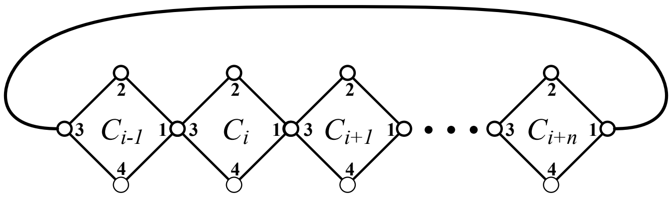


Fig. 1. Closed chain of contours of $1 \times n$ size

In the discrete case, each contour is divided into individual nods containing particles. The particles move along the contour from one cell to another according to a given direction. The rules for their movement can be analogous to traffic models such as the Nagel-Schreckenberg model or two-dimensional traffic models on a toroidal lattice (e.g., the BML model).

Each loop has one particle that occupies a node and can move at each discrete time instant $t = 1, 2, 3, \ldots$.

If at time t the particle is in the node of the contour with number i, then at the moment of time $t+1$ it appears in the node with number $i+1$ (modulo 4) – counterclockwise movement, or $i-1$ (modulo 4) – clockwise movement, if there are no obstacles for its movement.

The particle is not moved if the node, into which the particle is trying to get, is currently occupied by a particle of the neighboring contour (Fig. 2).

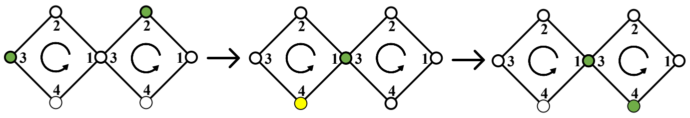


Fig. 2. Movement of particles along two contours. The particle that has travelled is highlighted in green, the particle waiting for a free node is highlighted in yellow

To resolve conflicts that arise when particles from adjacent contours simultaneously attempt to occupy the same common cell, various rules are applied:

1. Left-priority rule. In case of a conflict, only the particle with the smallest index i moves. This rule ensures deterministic conflict resolution.

2. Fair resolution rule. Each of the conflicting particles has an equal probability $p = 0.5$ of gaining priority for movement. This introduces a quasi-deterministic element into the system.

3. Even-odd rule. Priority is determined by the parity of the contour index. If index i is even, contour C_i has priority over adjacent contours. If index i is odd, contour C_i does not have priority.

An important concept in contour networks is collapse, which is defined as a state of the system where no particle from the set of contours moves, starting from some time $t \geq t_o$, where t_o is the model's start time.

In [11], the concept of a contour network's spectrum is introduced. For a deterministic contour network, if the system's state repeats, it will continue to repeat periodically thereafter. Such a cyclic trajectory in the set of system states is called a spectral cycle. The average velocity of a particle or cluster is defined as the ratio of the distance traveled by the particle/cluster during a period to the duration of that period. A spectral pair is a spectral cycle associated with a vector of average cluster velocities. The initial state of the system plays a key role in determining which spectral cycle will be realized, and consequently, what the average velocities will be. The set of all possible spectral pairs for given system parameters and various initial states forms the spectrum of the contour network.

3 Chainmail type contour network

In this section, a two-dimensional contour network known as a chainmail is introduced. It represents a discrete-time and discrete-state dynamical system. The carrier of this system is a set of $4mn$ contours of size $2m \times 2n$. Each cell of a contour is common for this contour and one of the four neighboring ones and has its own number in each of them.

To formalize the concept of a mail chain, we introduce a toroidal system of contours called a closed chainmail (Fig. 3). In such a system, for any contour $C_{i,j}$, where $i = 1,\dots,2m$ and $j = 1,\dots,2n$, the neighboring contours are $C_{i,j+1}$, $C_{i,j-1}$, $C_{i+1,j}$ and $C_{i-1,j}$ (taking into account modulo operations $2m$ or $2n$ for indices i and j respectively.

A key aspect of chainmail analysis is the concept of diagonals. Let k denote the greatest common divisor (GCD) of m and n. Then the entire set of contours of the chainmail can be divided into k non-overlapping subsets, called diagonals. An important property of these diagonals is that if at least one contour in a given subset is in a state of collapse, then the entire subset (diagonal) is also in a state of collapse. Each diagonal contains the same number of contours. Since the total number of contours in the mail chain is $4mn$, each diagonal contains $L = \frac{4mn}{k}$ contours.

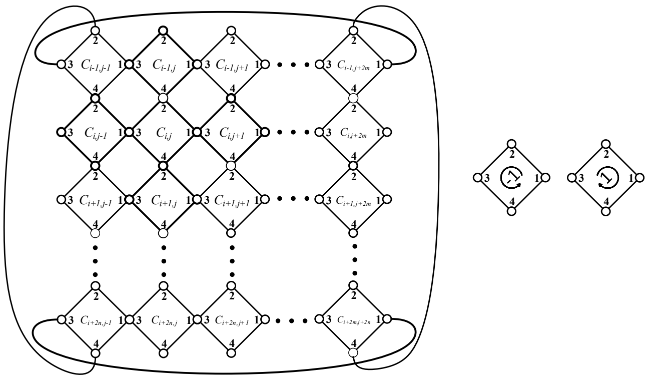


Fig. 3. Closed chainmail of size $2m \times 2n$ and examples of clockwise (1) and counterclockwise (-1) movement along contour

The state space of the chainmail is defined by the current positions of particles on all contours. The current state of the chainmail can be represented as a matrix S, where each element $S_{i,j}$ corresponds to contour $C_{i,j}$ and denotes the node number $c_{i,j}$ on that contour where a particle is currently located.

$$S = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1\ 2n} \\ c_{21} & c_{22} & \cdots & c_{2\ 2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{2m\ 1} & c_{2m\ 2} & \cdots & c_{2m\ 2n} \end{pmatrix}$$

The movement of particles along the contour can one-directional or co-directional. In one-directional movement, all particles move in the same direction clockwise or anti-clockwise. In the co-directional movement, if a particle moves clockwise on a contour, then in neighboring contours the movement is counterclockwise and vice versa.

4 Combinatorial analysis of diagonals and velocity configurations of the chainmail

In this section, a detailed combinatorial analysis of the properties of a contour network of the chainmail type is carried out, focusing on the characteristics of the diagonals and the spectrum of average velocities. These properties are crucial for understanding the dynamics of the system and its optimization.

4.1 Number of diagonals and contours per diagonal

Consider chainmail of size $2m \times 2n$. The number of diagonals k in such a chainmail is defined as the greatest common divisor (GCD) of $2m$ and $2n$, i.e., $k = GCD(2m,2n)$. Each diagonal contains the same number of contours. The total number of contours in the chainmail is $4mn$. Therefore, the number of contours on each diagonal is equal to $\frac{4mn}{GCD(2m,2n)}$.

Example 1. Let's illustrate the calculation of the number of diagonals k and the number of contours per diagonal L for a 4×4 chainmail. In this case, $2m = 4$ and $2n = 4$.

We calculate the greatest common divisor $k = GCD(m,n) = GCD(2,2) = 2$.

Since the entire set of contours is divided into k diagonals, and k =2, the number of diagonals is 2.

Total number of contours on each diagonal $L = \frac{4mn}{k} = \frac{16}{2} = 8$. Thus, each diagonal contains $L = 8$ contours (Fig. 4).

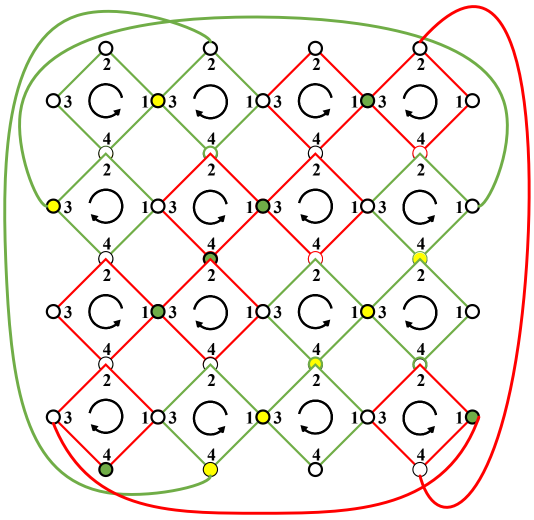


Fig. 4. Diagonal on a 4×4 closed chainmail. The contours belonging to the two different diagonals are highlighted in green and red. Nodes in the state of waiting/collapse are highlighted in yellow. Green in the state of free movement

The dependence of the number of diagonals on the chainmail size is presented in Table 1. From the Table 1, it is evident that the number of diagonals increases linearly with $m = n$ and exhibits symmetry with respect to m and n. A graphical representation of this dependence is shown in Figure 5.

Table 1

Dependence of the number of diagonals on the chainmail dimension $2m \times 2n$

2m\2n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
6	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3	1
8	1	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4
10	1	1	1	1	5	1	1	1	1	5	1	1	1	1	5	1
12	1	2	3	2	1	6	1	2	3	2	1	6	1	2	3	2
14	1	1	1	1	1	1	7	1	1	1	1	1	1	7	1	1
16	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	8
18	1	1	3	1	1	3	1	1	9	1	1	3	1	1	3	1
20	1	2	1	2	5	2	1	2	1	10	1	2	1	2	5	2
22	1	1	1	1	1	1	1	1	1	1	11	1	1	1	1	1
24	1	2	3	4	1	6	1	4	3	2	1	12	1	2	3	4
26	1	1	1	1	1	1	1	1	1	1	1	1	13	1	1	1
28	1	2	1	2	1	2	7	2	1	2	1	2	1	14	1	2
30	1	1	3	1	5	3	1	1	3	5	1	3	1	1	15	1
32	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	16

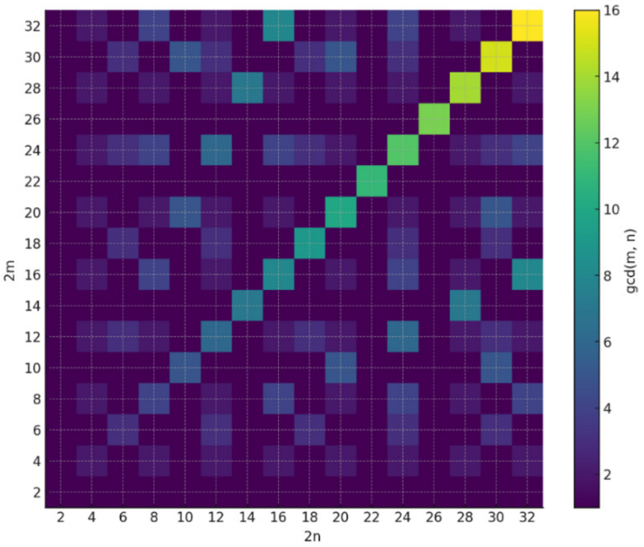


Fig. 5. Dependence of the number of diagonals on the size of the chainmail dimension $2m \times 2n$

Thus, the range of values for the number of diagonals k is defined as follows:

$$1 \le k \le \min(m,n).$$

The minimum number of diagonals $k = 1$ is achieved when m and n are coprime. The maximum number of diagonals $k = \min(m,n)$ is achieved when $m = n$.

Let's consider the dependence of the length (number of contours) L in a diagonal on the chainmail size. From Table 2, it is evident that the number of contours on a diagonal when $m = n$ is equal to $2m + 2n$. A graphical representation of this dependence is shown in Figure 6.

The minimum diagonal size $L_{\min} = 4mn$ is observed when m and n are coprime. The maximum diagonal size $L_{\max} = 4 \cdot \min(m,n)$ is achieved when $m \bmod n = 0$ or $n \bmod m = 0$.

Table 2

Dependence of the number of contours on a diagonal on the chainmail dimension 2m×2n

2m\2n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
2	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
4	8	8	24	16	40	24	56	32	72	40	88	48	104	56	120	64
6	12	24	12	48	60	24	84	96	36	120	132	48	156	168	60	192
8	16	16	48	16	80	48	112	32	144	80	176	48	208	112	240	64
10	20	40	60	80	20	120	140	160	180	40	220	240	260	280	60	320
12	24	24	24	48	120	24	168	96	72	120	264	48	312	168	120	192
14	28	56	84	112	140	168	28	224	252	280	308	336	364	56	420	448
16	32	32	96	32	160	96	224	32	288	160	352	96	416	224	480	64
18	36	72	36	144	180	72	252	288	36	360	396	144	468	504	180	576
20	40	40	120	80	40	120	280	160	360	40	440	240	520	280	120	320
22	44	88	132	176	220	264	308	352	396	440	44	528	572	616	660	704
24	48	48	48	48	240	48	336	96	144	240	528	48	624	336	240	192
26	52	104	156	208	260	312	364	416	468	520	572	624	52	728	780	832
28	56	56	168	112	280	168	56	224	504	280	616	336	728	56	840	448
30	60	120	60	240	60	120	420	480	180	120	660	240	780	840	60	960
32	64	64	192	64	320	192	448	64	576	320	704	192	832	448	960	64

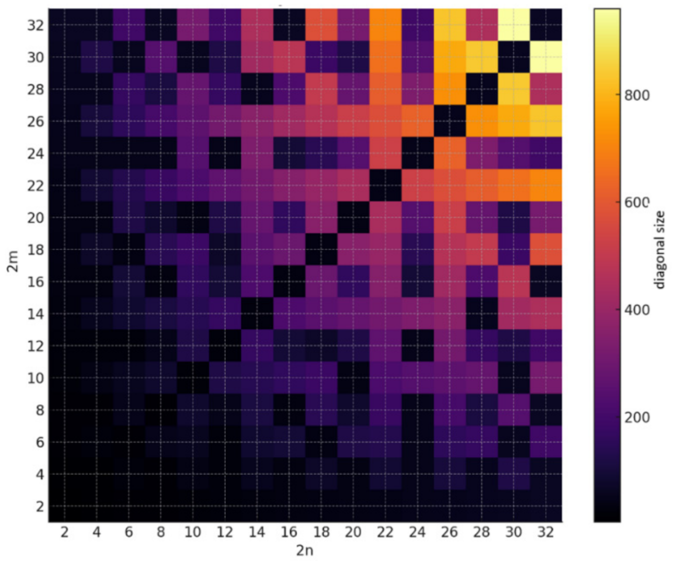


Fig. 6. The size of one diagonal (number of contours) in the chainmail $2m \times 2n$

4.2 Spectrum of chainmail average velocities

Let's consider the theorem on the spectrum of average velocities on a diagonal for a closed chainmail with co-directional movement.

Theorem 1 [12]. *For any closed chainmail with co-directional movement, the following is true. For any set of integers numbers* $v_1, \ldots, v_k$*, such that* $0 \le v_i \le \frac{mn}{k}$*, an initial state can be set such that the average velocity of a particle on any contour on the i-th diagonal is equal to* $\frac{v_i k}{mn}$, $i = 1, \ldots, k$.

According to Theorem 1, the cardinality of the set of average velocities on each diagonal is equal to $|V_k| = \frac{mn}{k} + 1$.

Example 2. Let's find the cardinality of the set of average velocities on diagonal k using the example of a closed chainmail of size 2×4 with co-directional movement. This configuration corresponds to a single diagonal of length $L = 8$:

$$|V_k| = \frac{mn}{k} + 1 = \frac{1 \cdot 2}{1} + 1 = 2 .$$

The cardinality of the set of all system states is $4^{2\times4} = 2^{16} = 65536$. Assuming that no more than 1 particle is located at a single node, 14068 valid states (21%) were found through computer generation.

Using simulation modeling of movement on a chainmail with the left-priority conflict resolution rule, the following distribution of average velocities $\bar{v}$ was obtained for each valid initial state:

1. $\bar{v} = 0$: 8 (0,05 %) initial states;
2. $\bar{v} = \frac{1}{2}$: 1602 (11,45 %) initial states;
3. $\bar{v} = 1$: 12458 (88,5%) initial states;

The obtained results correspond to the possible velocity spectra indicated in Theorem 1.

Let's consider the dependence of the cardinality of the set of average velocities $|V_k|$ on the chainmail dimension $2m \times 2n$. From Table 3, it is evident that the cardinalities of the set of average velocities depend linearly on $m = n$ and exhibit symmetry. The main order of growth is almost quadratic for coprime m and n. A graphical representation of this dependence is shown in Figure 7.

Table 3

Dependence of the cardinality of the set of average velocities $|V_k|$ on the chainmail dimension $2m \times 2n$

2m\2n	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**	**32**
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
4	3	3	7	5	11	7	15	9	19	11	23	13	27	15	31	17
6	4	7	4	13	16	7	22	25	10	31	34	13	40	43	16	49
8	5	5	13	5	21	13	29	9	37	21	45	13	53	29	61	17
10	6	11	16	21	6	31	36	41	46	11	56	61	66	71	16	81
12	7	7	7	13	31	7	43	25	19	31	67	13	79	43	31	49
14	8	15	22	29	36	43	8	57	64	71	78	85	92	15	106	113
16	9	9	25	9	41	25	57	9	73	41	89	25	105	57	121	17
18	10	19	10	37	46	19	64	73	10	91	100	37	118	127	46	145
20	11	11	31	21	11	31	71	41	91	11	111	61	131	71	31	81
22	12	23	34	45	56	67	78	89	100	111	12	133	144	155	166	177
24	13	13	13	13	61	13	85	25	37	61	133	13	157	85	61	49
26	14	27	40	53	66	79	92	105	118	131	144	157	14	183	196	209
28	15	15	43	29	71	43	15	57	127	71	155	85	183	15	211	113
30	16	31	16	61	16	31	106	121	46	31	166	61	196	211	16	241
32	17	17	49	17	81	49	113	17	145	81	177	49	209	113	241	17

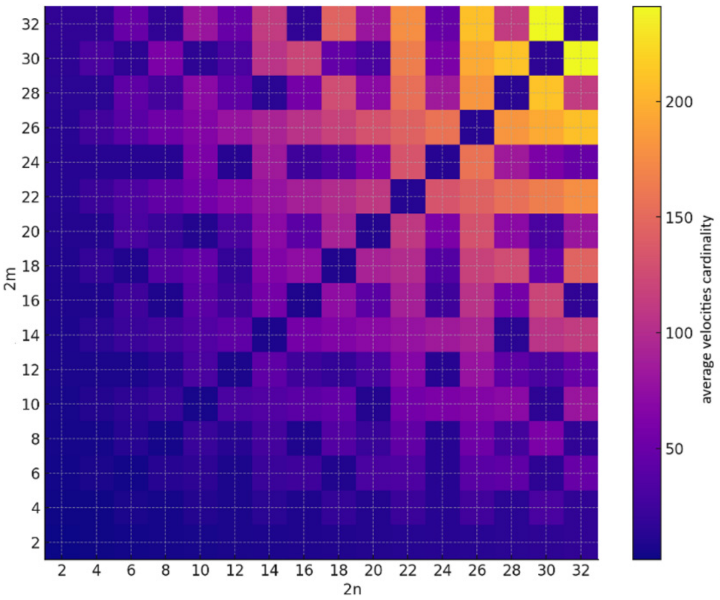


Fig. 7. The cardinality of the set of average velocities on the chainmail dimension $2m \times 2n$

Thus, the main formula for calculating the cardinality of the set of average velocities $|V_k|$ from the chainmail dimension $2m \times 2n$ on each diagonal k is defined as:

$$|V_k| = \frac{mn}{k} + 1 .$$

The minimum value of cardinality $|V_k|_{\min} = \min(m,n)+1$ is achieved when $m \bmod n = 0$ or $n \bmod m = 0$. The maximum value of cardinality $|V_k|_{\max} = mn+1$ is achieved when *m* and *n* are coprime.

5 Analysis of the influence of chainmail topology on the on the velocity spectrum

This section presents an empirical study of the influence of particle movement direction along contours on the average velocity of movement in a chainmail type contour network. The aim of the study is to identify correlations between movement direction configurations and achieved average velocities, as well as to verify the correspondence of empirical data with theoretical predictions, particularly Theorem 1 on the spectrum of average velocities.

Methodology of the study

The study was conducted for a chainmail of size $2m \times 2n$, where $2m = 4$ and $2n = 4$. For this configuration, $m = 1$, $n = 4$, and the greatest common divisor $\mathrm{GCD}(m,n) = \mathrm{GCD}(1,4) = 1$. This means that the chainmail consists of 1 diagonal, containing 16 contours. The total number of contours in the chainmail is $4mn = 4 \times 2 \times 2 = 16$. The contour numbers in the chainmail forming a diagonal are given by the matrix:

$$\begin{pmatrix} 16 & 3 & & 7 & 8 & 11 & 12 & 15 \\ 1 & 2 & 5 & & & 10 & 13 & 14 \end{pmatrix}_{2\times 8}$$

The cardinality of the set of all possible states of such a system is $4^{2\times 8} = 2^{32}$. Considering that only about 20% of states are valid (by analogy with a smaller chainmail) a full enumeration and simulation of all these states to determine the distribution of the velocity spectrum is computationally infeasible. Therefore, it was decided to investigate the influence of movement direction by analyzing the system's behavior for several representative initial particle configurations corresponding to various possible average velocities predicted by Theorem 1.

The following initial particle configurations were chosen, corresponding to average velocities $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$:

1. For average velocity 0:

$$S_0 = \begin{pmatrix} 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3 \\ 3\ 2\ 3\ 2\ 3\ 2\ 3\ 2 \end{pmatrix}_{2\times 8}$$

2. For average velocity $\frac{1}{4}$:

$$S_{\frac{1}{4}} = \begin{pmatrix} 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3 \\ 3\ 2\ 3\ 2\ 3\ 2\ 1\ 0 \end{pmatrix}_{2\times 8}$$

3. For average velocity $\frac{1}{2}$:

$$S_{\frac{1}{2}} = \begin{pmatrix} 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3 \\ 3\ 2\ 3\ 2\ 1\ 0\ 1\ 0 \end{pmatrix}_{2\times 8}$$

4. For average velocity $\frac{3}{4}$:

$$S_{\frac{3}{4}} = \begin{pmatrix} 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3 \\ 3\ 2\ 1\ 0\ 1\ 0\ 1\ 0 \end{pmatrix}_{2\times 8}$$

5. For average velocity 1:

$$S_1 = \begin{pmatrix} 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3 \\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \end{pmatrix}_{2\times 8}$$

Clockwise movement was denoted as 1, and counter-clockwise as -1. We generate matrices of size $2m \times 2n$, specifying the direction of movement in the corresponding contours of the chainmail. For each selected initial state, $2^{2\times 8} = 2^{16} = 65536$ possible configurations of the directions of movement were considered.

Using simulation modeling, the average velocity of movement was calculated for each given initial state and all possible configurations of movement directions along the chainmail contours. The simulation was conducted over 1000 iterations. The obtained average velocities $\bar{v}$ were grouped with a step of 0.05, and for each range, the number of direction matrices resulting in a velocity within that range was counted. The results are presented in Table 4.

Table 4

Distribution of average velocities depending on the initial state of the chainmail and direction configuration

Initial chainmail state velocity	**Velocity range**	**Average velocity in range**	**Number of direction matrices in range**
0	[0– 0,05)	0	2
	[0,25 – 0,3)	0.25	56
	[0,5 – 0,55)	0.5	652
	[0,55 – 0,6)	0.59338	408
	[0,6 – 0,65)	0.640441	832
	[0,65 – 0,7)	0.672881	8776
	[0,7 – 0,75)	0.73392	13880
	[0,75 – 0,8)	0.780731	2286
	[0,8 – 0,85)	0.819058	4398
	[0,85 – 0,9)	0.869021	1632
	[0,9 – 0,95)	0.907	8
	[0,95 – 1]	0.995876	32604
$\frac{1}{4}$	[0,25 – 0,3)	0.25	16
	[0,45 – 0,5)	0.487482	168
	[0,5 – 0,55)	0.513335	1238
	[0,55 – 0,6)	0.575136	1188
	[0,6 – 0,65)	0.636016	1602
	[0,65 – 0,7)	0.672653	9263
	[0,7 – 0,75)	0.737832	17820
	[0,75 – 0,8)	0.779048	2112
	[0,8 – 0,85)	0.820541	3509
	[0,85 – 0,9)	0.867652	1515
	[0,9 – 0,95)	0.907079	38
	[0,95 – 1]	0.996149	27067
$\frac{1}{2}$	[0,3 – 0,35)	0.336328	64
	[0,35 – 0,4)	0.375	16
	[0,4 – 0,45)	0.42825	16
	[0,45 – 0,5)	0.494137	752
	[0,5 – 0,55)	0.514935	2215
	[0,55 – 0,6)	0.571746	1779
	[0,6 – 0,65)	0.633409	2378
	[0,65 – 0,7)	0.672204	9938
	[0,7 – 0,75)	0.739932	20794
	[0,75 – 0,8)	0.779365	1673
	[0,8 – 0,85)	0.821316	2789
	[0,85 – 0,9)	0.865962	1382
	[0,9 – 0,95)	0.907362	47
	[0,95 – 1]	0.996379	21693

$\frac{3}{4}$	[0,25 – 0,3)	0.25	64
	[0,3 – 0,35)	0.33645	129
	[0,35 – 0,4)	0.375	64
	[0,4 – 0,45)	0.428321	28
	[0,45 – 0,5)	0.496248	1542
	[0,5 – 0,55)	0.512895	3861
	[0,55 – 0,6)	0.569845	2484
	[0,6 – 0,65)	0.632881	2733
	[0,65 – 0,7)	0.672124	9921
	[0,7 – 0,75)	0.741252	21910
	[0,75 – 0,8)	0.779485	1247
	[0,8 – 0,85)	0.822359	2290
	[0,85 – 0,9)	0.865558	1286
	[0,9 – 0,95)	0.907587	46
	[0,95 – 1]	0.996625	17931
1	[0– 0,05)	0	2
	[0,25 – 0,3)	0.25	239
	[0,3 – 0,35)	0.336387	248
	[0,35 – 0,4)	0.375	128
	[0,4 – 0,45)	0.428381	42
	[0,45 – 0,5)	0.497192	2232
	[0,5 – 0,55)	0.511239	5401
	[0,55 – 0,6)	0.569829	2858
	[0,6 – 0,65)	0.632487	2827
	[0,65 – 0,7)	0.672318	9796
	[0,7 – 0,75)	0.741949	21871
	[0,75 – 0,8)	0.778399	979
	[0,8 – 0,85)	0.822896	1890
	[0,85 – 0,9)	0.866669	1608
	[0,9 – 0,95)	0.907612	49
	[0,95 – 1]	0.996728	15363

5.1 Analysis of results

We will assume that the range [0.95 – 1] corresponds to an average velocity of 1 since a slightly lower value corresponds to simulation modelling accuracy. From Table 4, it is evident that for all considered initial states, none of the movement direction configurations resulted in an average velocity lower than that corresponding to the statement of Theorem 1.

The exceptions are the states:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

for the initial state with velocity 0, which are symmetric states of co-directional movement, and the states

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for the initial state with velocity 1, which are symmetric states of one-directional movement.

This confirms the theoretical predictions regarding the minimally achievable velocities in the system.

The distributions of average velocities for various initial states (Fig. 8-12) demonstrate similar patterns, indicating the stability of the system's dynamics to variations in initial conditions.

Of particular interest are two ranges of average velocities that contain the largest number of movement direction matrices: [0.7,0.75) and [0.95,1]. For a deeper understanding of the influence of movement direction, an analysis of the prevailing movement direction for each contour within these ranges was conducted.

The prevailing direction was determined by subtracting the total number of -1 values (counter-clockwise) from the total number of 1 values (clockwise) for each cell of the state matrix.

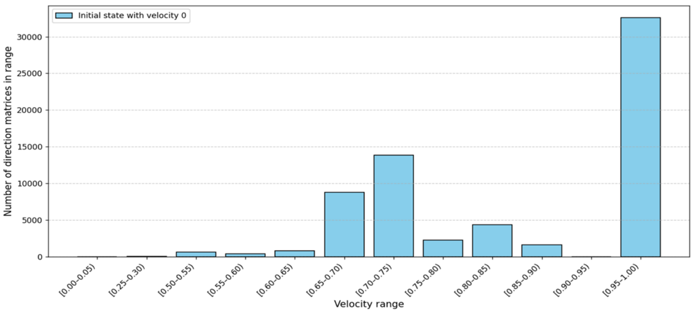


Fig. 8. The distributions of average velocities for the initial state with velocity 0

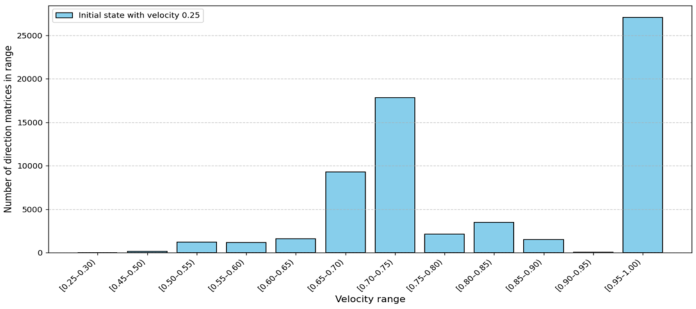


Fig. 9. The distributions of average velocities for the initial state with velocity $\frac{3}{4}$

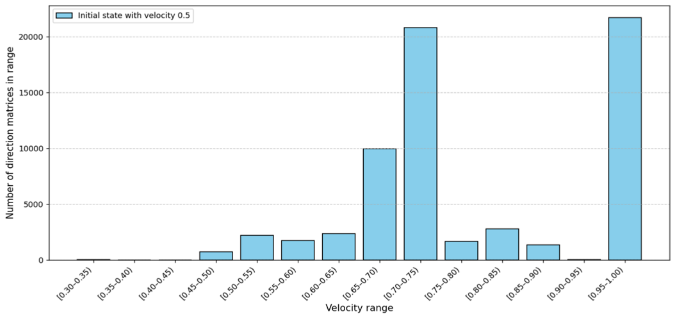


Fig. 10. The distributions of average velocities for the initial state with velocity $\frac{1}{2}$

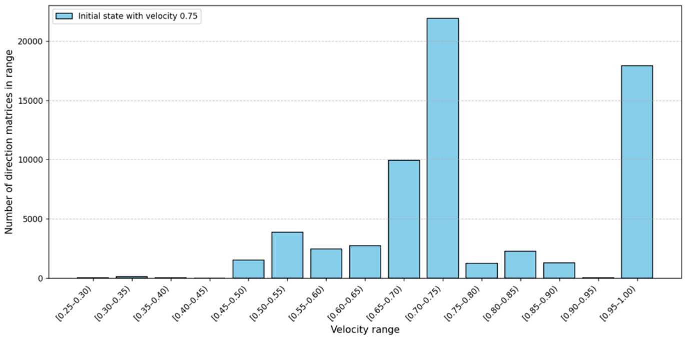


Fig. 11. The distributions of average velocities for the initial state with velocity $\frac{3}{4}$

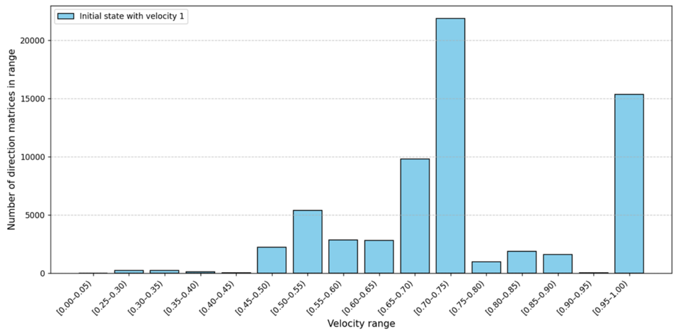
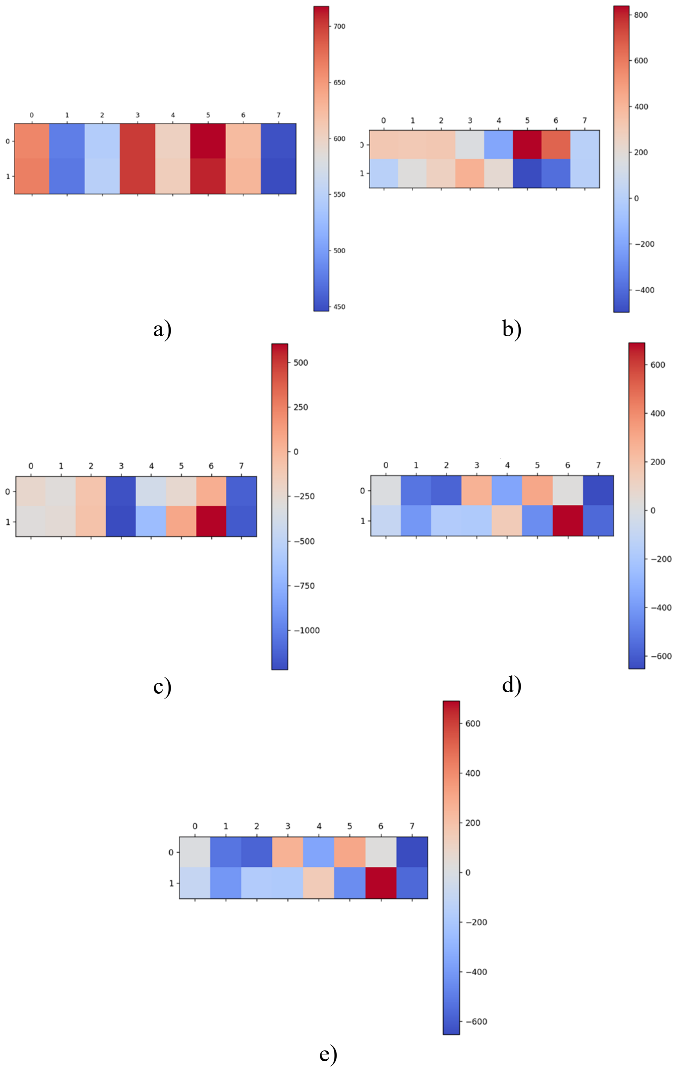


Fig. 12. The distributions of average velocities for the initial state with velocity 1

The results are visualized in Figure 13-14, where the red palette corresponds to a predominance of clockwise movement, and the blue palette to counter-clockwise.



a) b) c) d) e)

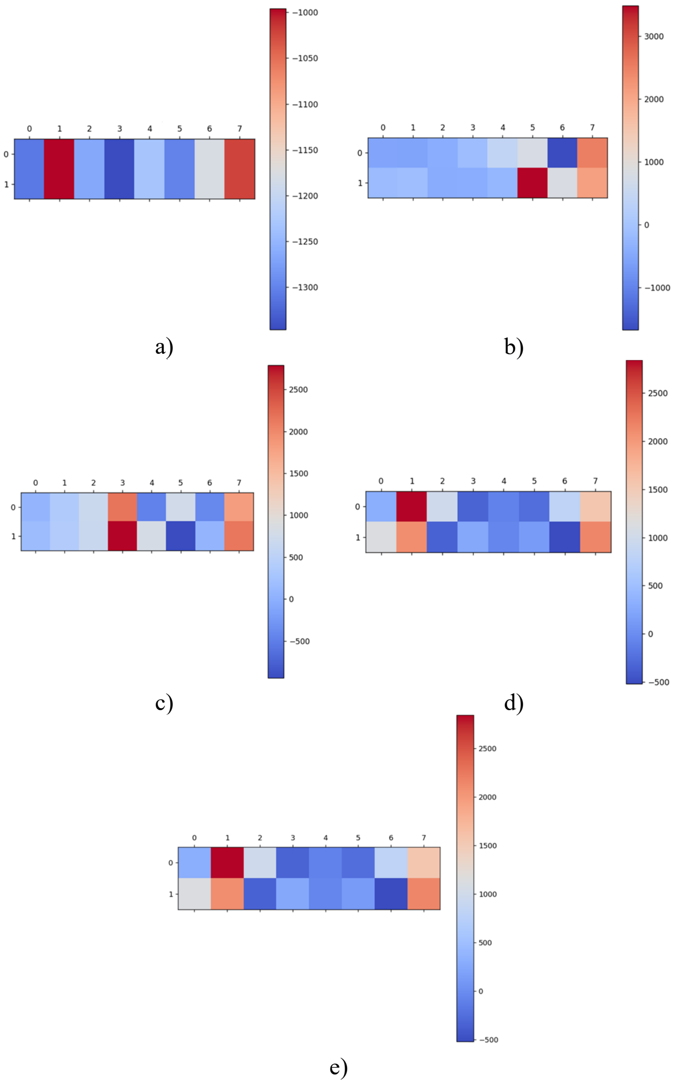
Fig. 13. Dominant direction of movement for each contour in the velocity range [0.7, 0.75) for the initial state with velocity a) b) $\frac{1}{4}$, c) $\frac{1}{2}$, d) $\frac{3}{4}$, e) 1

Let's analyze the range [0.7,0.75).

Figure 13 a) (initial state with velocity 0) shows a characteristic tendency to maintain the direction of movement horizontally is observed. This means that in horizontally arranged contours, particles predominantly move in one direction, which contributes to maintaining a relatively high average velocity.

Figures 13 b), c) (initial states with velocities 0.25 and 0.5) shows a tendency towards unidirectional movement is noted, with interspersed groups of four contours exhibiting co-directional movement. This indicates that to achieve average velocities in this range, the system can utilize a combination of unidirectional and co-directional movement patterns.

Figures 13 d), e) (initial states with velocities 0.75 and 1) shows a tendency towards co-directional movement across the entire chainmail is observed. This suggests that to maintain high average velocities, the system optimizes particle movement such that adjacent contours move in opposite directions, minimizing conflicts and ensuring a smoother flow.



a) b) c) d) e)

Fig. 14. Dominant direction of movement for each contour in the velocity range [0.95, 1] for the initial state with velocity a) b) $\frac{1}{4}$, c) $\frac{1}{2}$, d) $\frac{3}{4}$, e) 1

Let's analyses the range [0.95,1]:

Figure 14 a) (initial state with velocity 0) shows a clear tendency to maintain the direction of movement horizontally is evident, similar to Figure 13 a), but with a more distinct pattern.

This confirms that horizontally-oriented unidirectional movement is a key factor for achieving high velocities, even with an initially low system velocity.

Figure 14 b), c), d) (initial states with velocities 0.25, 0.5, 0.75, and 1) shows an intensification of the tendency towards co-directional movement across the entire chainmail is observed. This indicates that to achieve the maximum possible average velocities, the system strives for global coordination of movement directions, where each contour interacts with its neighbors in a way that ensures optimal particle flow.

Based on the conducted analysis, the following assumptions can be formulated regarding a chainmail with co-directional movement:

1. In a collapse state, where particle movement is practically absent, a significant increase in average velocity and transition to free movement is possible. This is achieved by maintaining the one-directional vertical movement direction and alternating it horizontally. This pattern effectively resolves potential conflicts and initiates movement.
2. For initial states with low velocity, it can be significantly increased by introducing areas with unidirectional movement. Creating such “flow” zones contributes to accelerating particles and improving the overall system dynamics.
3. For initial states with already high velocity, co-directional movement across the entire chainmail ensures its stable maintenance. This indicates that global coordination of movement directions is an optimal strategy for maximizing the system’s throughput.

The obtained data have significant practical value and can be used in modeling real-world road networks. The application of the identified movement direction patterns can contribute to increasing their throughput, reducing congestion, and optimizing traffic flows in urban environments.

6 Conclusion

In this paper, a comprehensive study of combinatorial and dynamical properties of contour networks of the chainmail type on the torus, which are discrete dynamical systems for modelling traffic flows, has been carried out. The main focus was on the analysis of the structure of diagonals, numerical determination of average velocity spectra, as well as to the study of the influence of the directionality of particle movement on the dynamics of the whole system.

On the basis of analytical formulae, the dependence of the number of diagonals, the length of diagonals and the cardinality of the set of average velocities on the dimensionality of the chainmail was established. Generalized relations describing the velocity spectrum depending on the network parameters are obtained. The concept of the state space of the chainmail is introduced, which allows to formalize the whole set of admissible configurations of particle placement in the system taking into account the limitations of the mutual position of particles in neighboring contours.

Simulation experiments have shown that the choice of directionality of movement on individual contours has a significant effect on the average velocity of the whole system. Numerical calculations revealed the characteristic regularities of velocity profile formation, including the formation of zones of one-directional and co-directional movement, which contribute to the optimization of the network capacity.

The developed methods of combinatorial analysis and optimization of directions of movement have high potential applicability in the design of real street and road networks [13], flow management in metropolises [14], as well as in the modelling of transport systems with high requirements to the stability and efficiency of flows [15].

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МЕТОД ОПТИМИЗАЦИИ СЕТИ БУСЛАЕВА ТИПА КОЛЬЧУГА НА ТОРЕ ПРИ ВАРИАЦИИ ТОПОЛОГИИ

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Аннотация

Работа посвящена исследованию и оптимизации контурных сетей Буслаева типа кольчуга на торе, которые представляют собой дискретные динамические системы со сложной топологией. Эти сети явно учитывают конфликты в общих узлах контуров, что делает их особенно подходящими для моделирования транспортных потоков в сложных городских дорожных сетях, где пересечения и перекрывающиеся транспортные потоки играют критическую роль в определении общей пропускной способности и устойчивости системы. В работе представлен комплексный комбинаторный анализ топологии сети типа кольчуга, включающий определение количества диагоналей, распределение контуров по диагоналям и вывод спектра средних скоростей частиц. Разработаны аналитические формулы для оценки числа допустимых состояний и допустимых конфигураций скоростей с учетом пространственных и топологических ограничений, присущих системе. Проведено имитационное исследование контурной сети типа кольчуга, демонстрирующее четкую корреляцию между конфигурациями направления движения и результирующими средними скоростями частиц. Предложен метод оптимизации топологии, позволяющий определить оптимальные конфигурации направления движения, которые максимизируют среднюю скорость системы и при этом снижают количество локальных конфликтов. Полученные результаты подтверждают теоретические предсказания относительно минимально достижимых скоростей и выявляют структурно устойчивые оптимальные конфигурации, значительно повышающие пропускную способность системы. Сформулированы практические рекомендации по применению разработанных моделей для оптимизации транспортных потоков в реальных городских улично-дорожных сетях, что открывает возможности для повышения эффективности перевозок в условиях сложных топологических и эксплуатационных ограничений.

Ключевые слова: *контурные сети Буслаева, дискретные динамические системы, моделирование транспортных потоков, комбинаторный анализ, спектр скоростей, оптимизация топологии*

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