ON PROBLEM OF PROPERTIES OPTIMIZING FOR VISCOELASTIC MATERIALS USING THE BEGLEY-TORVIK EQUATION

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Recently, fractional calculus has been the focus of attention of many researchers in the field of science and technology, since a more detailed study of physical processes leads to the need to complicate the mathematical models that describe them, and, consequently, to the study of the behavior of solutions of differential equations containing, along with " ordinary", or "classical", derivative, also fractional. Processes of this kind can include: studies of continuous media with memory, fluid filtration in media with fractal geometry, physical aspects of stochastic transfer and diffusion, mathematical models of a viscoelastic body, models of damped oscillations with fractional damping (for example, vibrations of rocks during earthquakes or vibrations nanoscale sensors), models of non-local physical processes and phenomena of a fractal nature; climate models, etc. The paper studies boundary value problems for the equation of motion of an oscillator with viscoelastic damping (the Begley-Torvik equation) in the case when the damping order is greater than zero but less than two. Such problems model many physical processes, in particular, the vibration of a string in a viscous medium, the change in the deformation-strength characteristics of polymer concrete under loading, etc. This paper is devoted to optimizing the parametric control of the Begley-Torvik model. A fundamentally new, efficient algorithm is proposed that allows estimating the parameters of a model of real material.

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Introduction

Fractional calculus is a mathematical field that deals with integrals and derivatives of arbitrary order. Although the concept dates back to 1695, it was only in the last century that the most impressive advances were made. In particular, over the past three decades, fractal theory [1], fractional differential equations have found applications in physics, signal processing, engineering, biological sciences and finance [2, 3, 4].

First of all, we note that fractional derivatives with respect to space can be used to model anomalous diffusions or dispersions, and fractional derivatives with respect to time can be used to model some processes with "memory".

It is known that it is expedient to model the stress-strain state of viscoelastic materials using fractional differential operators leading to differential equations with fractional derivatives.

Of particular interest are second-order differential equations with fractional derivatives in lower terms. Such equations, in particular, are used to describe the vibration of a string, taking into account friction in a medium with fractal geometry, or to simulate changes in the deformation and strength characteristics of polymer concrete (polymer concrete is a type of concrete mixture made on the basis of one of the synthetic resins) under the action of loads.

On methods of using fractional calculus in problems of viscoelastic media modeling

In this paper, samples of polymer concrete based on polyester resin were taken for research. As a polyester resin, polyesters based on diane and dichloride-1,1-dichloro-2,2 di (n-carboxyphenyl) ethylene. Although all polyester resins are similar, a wide range of mechanical properties can always be achieved in their production by changing the basic constituents and their proportions. In our case, polymer concrete is represented as a set of mineral aggregate granules in a viscous medium of polyester resin.

When modeling the deformation-strength characteristics of polymer concrete, it can be represented as a set of solid filler granules located in a viscoelastic medium. Then [3], the transverse motion of the filler granule under the action of loads (applying an external force) can be described by the equation (this equation is called the Begley-Torvik equation [3]) of the fractal (fractional) oscillator:

$$m\cdot u''(x)+\nu\cdot D^{\alpha}_{ox}u(x)+k\cdot u(x)=\zeta(x), \quad (1)$$

where $u(x)$ – granule displacement, $x\in[0;l]$,

m – filler granule weight,

ν – resin viscosity modulus,

k – resin stiffness modulus;

α – medium viscoelasticity parameter,

ζ – external force.

Definition: $D^{\alpha}u(x)$ – is the fractional differential operator of order $\alpha\in[0;2]$ in Riemann-Liouville sense, i.e.:

1) for $\alpha\in[0;1]$

2)

$$D^{\alpha}u(x)=\frac{d}{dx}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^x\frac{u(\tau)d\tau}{(x-\tau)^{\alpha}}\right),$$

where $\Gamma(x)$ – is the Euler's gamma – function;

3) for $\alpha\in[1;2]$ we have

$$D^{\alpha}_{ox}u(x)=\frac{d^2}{dx^2}\left(\frac{1}{\Gamma(2-\alpha)}\int_0^x\frac{u(\tau)d\tau}{(x-\tau)^{\alpha-1}}\right).$$

Note [4], that for $\alpha=1$, the equation (1) transforms to the well-known equation

$$m\cdot u''(x)+\nu\cdot u'(x)+k\cdot u=\zeta(x), \quad (2)$$

which describes the movement (in asphalt concrete) of a granule of mass m under the action of a load $\zeta(x)$ from moving vehicles, which is widely used in road construction.

It should be noted [5]-[7] that rutting of roads can be formed with any type of road surface.

For $\alpha=0.7$ the equation

$$m\cdot u''(x)+\nu\cdot D^{0.7}_{ox}u(x)+k\cdot u=\zeta(x), \quad (3)$$

beings [2] a good constitutive model for elastomeric bearings (elastomeric bearings are currently used as insulating bearings to protect building bridges, etc. from earthquakes).

This paper is devoted to the optimization of the parametric control of the Begley-Torvik model. A fundamentally new, efficient algorithm has been proposed that allows estimating the parameters of a model of real materials.

First of all, we give the following well-known statement [8, 14].

Theorem 1. Solution of the boundary value problem

$$u''(x)+cD^{\alpha}u(x)+\lambda u(x)=0; \quad (4)$$

$$u(0)=0\ u(l)=0. \quad (5)$$

can be found using a sequence of recurrent kernels and written out as a power series

$$u(x)=x+\sum_{n=1}^{\infty}(-1)^n\sum_{m=0}^{n}\frac{\binom{n}{m}c^m\lambda^{n-m}x^{2n+1-m\alpha}}{\Gamma(2n+2-m\alpha)} \quad (6)$$

The eigenvalues are found as solutions to the implicit equation:

$$1=\sum_{n=1}^{\infty}(-1)^{n+1}\sum_{m=0}^{n}\frac{\binom{n}{m}c^m\lambda^{n-m}}{\Gamma(2n+2-m\alpha)}1^{2n+1-m\alpha}. \quad (7)$$

Relation (7) will be used below to determine the order of the fractional derivative.

Method for determining the parameters of the Begley-Torvik model

It is known [3] that to model the deformation-strength characteristics of viscoelastic materials, the equation

$$\sigma(t)=E_1D^{\beta}\varepsilon(t), \quad (8)$$

where $\sigma(t)$ – the stress, $\varepsilon(t)$ – the deformation, E_1 and $0<\beta<1$ – parameters of the matherial. Here,

$$D^{\beta}f(t)=\frac{1}{\tilde{A}(1-\beta)}\int_0^t\frac{f'(\tau)}{(t-\tau)^{\beta}}d\tau \quad (9)$$

fractional Caputo derivative, of order β, to be determined.

In numerous publications of the last ten years, the problem of identifying the parameters of fractional models is mainly solved at the theoretical level, for example, using spectral analysis methods. As noted in publications, in particular [9] V.P. Radchenko E.N. Ogorodnikov L. G. Ungarova, in the papers of T. S. Aleroev [8], [10], the model parameters are determined based on several characteristic points obtained in the experiment by substituting the strain values into the analytical solutions of the corresponding problem.

In this paper, the same technique is used to determine the order of the fractional derivative in the problem for the Begley-Torvik equation

$$u''(x) + cD^{\alpha}u(x) + \lambda u(x) = 0; \quad (10)$$

$$u(0) = 0\ u'(l) = 1; \quad (11)$$

here $D^{\alpha}u(x)$ – is the fractional differential operator of order $\alpha \in [0;2]$.

In [10],[13], solution (10) -(11) was calculated using a sequence of recurrent kernels and written out as a power series,

$$u(x) = x - \sum_{n=1}^{\infty}(-1)^n \sum_{m=0}^{n} \frac{\binom{n}{m} c^m \lambda^{n-m}}{\Gamma(2n+2-m\alpha)} x^{2n+1-m\alpha} \quad (12)$$

In order to shade the main ideas of this technique, we first of all dwell on the paper [10],[13], which describes a technique for determining the order of the fractional derivative.

In this paper, to determine the parameter β, in (8) it is assumed that the tension of the material is given linearly.

$$\varepsilon(t) = kt \quad (13)$$

Taking into account the well-known formula

$$D^{\beta}t = \frac{t^{1-\beta}}{A(2-\beta)}$$

we obtain

$$\sigma(t) = \frac{kE_1}{\Gamma(2-\beta)} t^{1-\beta} = \frac{k_1E_1}{\Gamma(2-\beta)}[\varepsilon(t)]^{1-\beta}.$$

Designated as $A = \frac{k^{\beta}E_1}{\Gamma(2-\beta)}$, we have

$$\sigma(t) = A[\varepsilon(t)]^{1-\beta}. \quad (14)$$

Thus, in this case, the stress depends on the strain according to the power law. To determine the β it suffices to know the results of two measurements *ε(t₁)* and *ε(t₂)*.

Of course, in nature, deformation is a far from linear function. But if it is possible to establish the parameter β for the case of linear loading, then, by the existence and uniqueness theorem for equation (8), it can be argued that this parameter is invariant and does not depend on the type of loading function.

In this paper we apply the same technique to determine the parameters of the Begley-Torvik model. To do this, we divide the interval (0,2) where the possible order of the fractional derivative is located into N equal parts (N is any natural number). And consider the tasks

$$u''(x) + cD^{\alpha\_i}u(x) + \lambda u(x) = 0;\ \ i = 1 \dots N; \quad (15)$$

$$u(0) = 0, u'(l) = 1; \quad (16)$$

Figure 1 shows graphs for solving these problems for various values of the order of fractional differentiation.

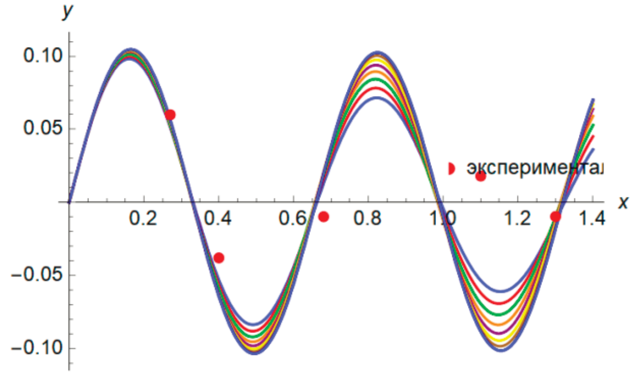

Figure 1. Problem solving graphs for 0 <α <2

First of all, we note:

a) that we proceed from the fact that the process under study is described by one of the tasks (15-16);

b) he existence and uniqueness theorem holds for these problems;

c) no matter how large N is, there is an interval (γ, β) included in (0,2), where the graphs for solving problems (15-16) do not intersect.

And now we will make the following important remark: when planning an experiment, take into account that we will be interested in experimental data or field measurements of the quantity $u(x)$ for $x \in (\gamma, \beta)$.

Now take any point x_0 form the interval (γ, β) and mark on the plane, where the graphs of solutions are shown, a point with coordinates $(x_0, u(x_0))$ ($u(x_0)$ we obtain from the experience).

The graph from figure 1, which passes through this point, will be the graph of the solution of the problem that this process models.

In order to more accurately determine the order of the fractional derivative, it is necessary to solve the following equation for α

$$u(x_0) = x_0 + \sum_{n=1}^{\infty}(-1)^n \sum_{m=0}^{n} \frac{\binom{n}{m} c^m \lambda^{n-m} {x_0}^{2n+1-m\alpha}}{\Gamma(2n+2-m\alpha)} \quad (17)$$

Generally speaking, this equation can have several roots (this does not contradict the existence and uniqueness theorem, since, generally speaking, the roots are found not of equation (17), but of the equation

$$u(x_0) = x_0 + \sum_{n=1}^{N}(-1)^n \sum_{m=0}^{n} \frac{\binom{n}{m} c^m \lambda^{n-m} {x_0}^{2n+1-m\alpha}}{\Gamma(2n+2-m\alpha)} \quad).$$

Of these, the desired root will be the one that induces the graph, which at the point xo deviates less than all other graphs from the point $(x_0, u(x_0))$.

For an approximate calculation of the sum of the series in (17), we take the first 80 terms. Thus, we solve numerically for α the following equation:

$$u(x_0) = x_0 + \sum_{n=1}^{80}(-1)^n \sum_{m=0}^{n} \frac{\binom{n}{m} c^m \lambda^{n-m} {x_0}^{2n+1-m\alpha}}{\Gamma(2n+2-m\alpha)}$$

Once again, we note that, for our polymer concrete c=1.2, $\lambda = 89$.

Note that there are practically no papers devoted to solving such equations, and therefore questions related to the accuracy of the solutions obtained remain open.

Results analysis

We test the above method on the example of the above polymer concrete.

In order to test the technique, we take the experimental data obtained in [10]. Values for polymer concrete samples based on PES (dian and dichloanhydride-1,1-dichloro-2,2-diethylene) are presented in Table 1.

Table 1

Experimental points for polymer concrete samples

x_i (c)	0,27	0,4	0,68	1,1	1,3	1,6
U_i	0,06	– 0,038	– 0,0098	0,018	– 0,0097	– 0,01

In order to determine the order of the fractional operator, consider the integral curves of the following Cauchy problem (Fig. 2)

$$my'' + cD_{ox}^{\alpha}y + \lambda y = 0,$$
$$y(0) = 0, \quad y'(0) = 1,$$

for different α, c, λ.

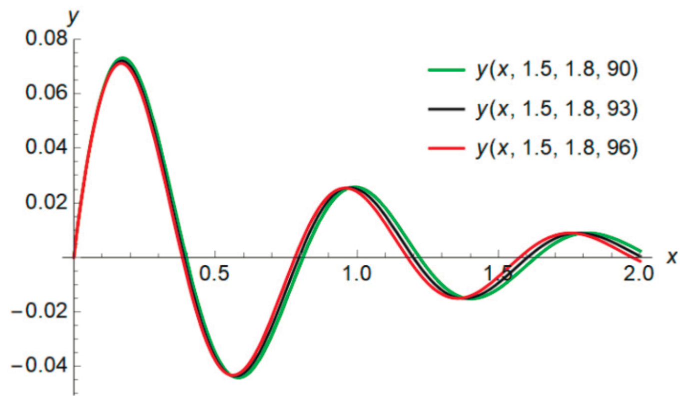

Fig. 2. Solutions of the Cauchy problem for the oscillator model with RT damping at α = 1.5, c = 1.8, λ = 90,93,96

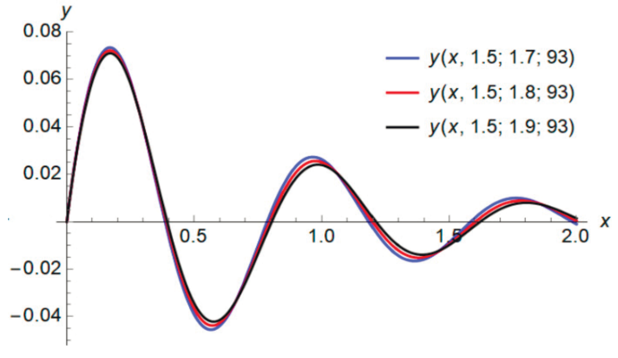

Fig. 3. Solutions of the Cauchy problem for the oscillator model with RT damping at α = 1.5, c = 1.7,1.8,1.9, λ = 93

Analysis of the graphs in Figures 2 and 3 shows that to determine the order of the fractional derivative, it is sufficient to choose an interval where these graphs do not intersect. Taking any point from this interval, we look at the value of the solution at this point. And this value is compared with experimental data. Experimental data [11] relevant for this case are given in Table 2.

Table 2

Experimental data

(0,46; 1,03)	(0,47; 1,02)	(0,48; 1,0198)	(0,49; 1,018)	(0,50; 1,0178)
(0,51; 0,93)	(0,52; 0,83)	(0,53; 0,5)	(0,54; 0,2)	(0,55; 0,18)
(0,56; – 0,01)	(0,57; – 0,3)	(0,58; – 0,61)	(0,59; –0,93)	(0,60; – 0,96)
(0,61; – 0,21)	(0,62; – 0,8)	(0,63; 0,21)	(0,64; 0,24)	(0,65; 0,301)
(0,66; 0,32)	(0,67; 0,38)	(0,68; 0,41)	(0,69; 0,43)	(0,70; 0,431)

To choose an interval where the graphs of the corresponding solutions do not intersect, we present the graphs of the solutions of the corresponding problems for different values of the order of fractional differentiation.

In accordance with our methodology, for the interval (γ, β) we can take the interval (0.35,0.45), and for the point x_0 we take x_o=0.4. And then the problem of finding α is reduced to solving the following equation

$$-0.038 = 0.4 + \sum_{n=1}^{80}(-1)^n \sum_{m=0}^{n} \frac{\binom{n}{m} 1.2^m 89^{n-m}(-0.038)^{2n+1-m\alpha}}{\Gamma(2n+2-m\alpha)} \quad (19)$$

Making the corresponding calculations using MATLAB, we get that α =1.4.

Thus, when it comes to samples of polymer concrete based on PES (dian and dichloanhydride-1,1-dichloro-2,2-diethylene), the equation of motion has the form

$$u''(x) + 1.2 \cdot D_{ox}^{1.4}u(x) + 89 \cdot u = 0 \quad (20)$$

The same problem was solved in the standard way in [10].

The unknown parameter α is located there minimizing the deviation of the theoretical curves from the experimental ones. That is, defining the deviation function by the least squares method

$$F(\alpha) = \sum_{i=1}^{N}(U_i - u(x_i, \alpha))^2$$

Here

$$u(x_i) = U_i, i = 1, \dots, N$$

several experimental points, and $u(x_i, \alpha)$- theoretical points calculated by formula (12).

The unknown parameter α is selected minimizing the deviation of the theoretical curves from the experimental ones.

The accuracy of the result obtained by this standard method depends on the sample size (the larger the sample size, the more accurate the result) and how accurate the U_i measurements obtained as a result of the experiment are.

The application of our technique leads to much more accurate results and does not require the costs of the experiment, which are necessary when implementing the standard technique given in [10].

And finally, as shown in Figures 1-3, the frequencies and amplitudes of the oscillations of the granules depend on the parameter α. In turn α, as mentioned, it characterizes the viscoelasticity of the medium (i.e., it completely depends on the physicochemical properties of the resin). So that the oscillation amplitudes do not go beyond the permissible limits, the intervals for changing α must also be clearly defined.

The method presented in this paper also solves this important problem. Since the optimal control of the parameter α makes it possible (using directed synthesis) to synthesize polyester resins with the required strength characteristics.

Let us give an example of using the developed technique in modeling viscoelastic properties and hysteresis damping of springs made of composite materials using fractional calculus [12]. In this case, the parameter on which the deflection of the spring depends (the spring test scheme is shown in Figure 4).

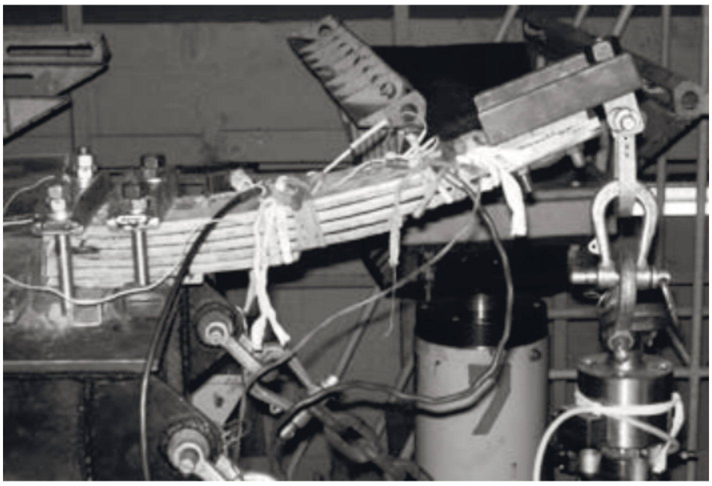

Fig. 4. Spring test is the order of the fractional derivative, the method of estimating which we have given above

Conclusion

The method of parametric identification of the order of a fractional derivative proposed in this paper allows us to solve a number of inverse problems in fractional calculus. In particular, to develop an effective method for optimizing the parametric control of the Begley-Torvik model.

References

1. D.V. Georgievsky, M.V. Shamolin, "Results of science and technology. Modern Mathematics and its Applications. Thematic reviews," Founders: All-Russian Institute of Scientific and Technical Information RAS, vol. 205, pp. 3-9, 2022.

2. C.G. Koh, J.M. Kelly, "Application of fractional derivatives to seismic analysis of base-isolated models," First published: February 1990 https://doi.org/10.1002/eqe.4290190207

3. R. Bagley, P. Torvik, "Fractional Calculus – a Different Approach to the Analysis of Viscoelastically Damped Structures," *Aiaa Journal - AIAA J*, no. 21, pp. 741-74, 1983.

4. A.M. Kirillov, "Modeling of energy exchange processes in the road surface – vehicle system, " 2015.

5. M.V. Yashina, P.I. Pospelov, I.A. Kuteinikov, "Influence of traffic flow distribution on the rutting of road surfaces," *Proceedings of the XIV all-russian multi-conference on management issues MKPU-2021*. Materials of the XIV multi-conference in 4 volumes. Vol. 4. Rostov-on-Don – Taganrog, 2021, pp. 107-110.

6. R.G. Abyshov, A.P. Buslaev, Yu.D. Kupriyanov, M.V. Yashina, "Distributed system for monitoring the maintenance of highways," *Vestnik MADI*, no. 1(24), 2011, pp. 79-85.

7. A.P. Buslaev, O.V. Zamotailov, P.I. Pospelov, M.V.Yashina, "On the restoration of pavement layers according to subsurface radar data (part 1)," *Bulletin of MADI (STU),* 2009, pp. 89-96.

8. T.S. Aleroev, H.T. Aleroeva, "Problems of Sturm-Liouville type for differential equations with fractional derivatives," In Volume 2 Fractional Differential Equations; Kochubei, A., Luchko, Y., Eds.; De Gruyter: Berlin, Germany; Boston, MA, USA, 2019, pp. 21-46.

9. E.N. Ogorodnikov, V.P. Radchenko, L.G. Ungarova, "Mathematical models of nonlinear viscoelasticity with operators of fractional integro-differentiation," *Bulletin of the Perm National Research Polytechnic University. Mechanics*, 2018, no. 2, pp. 147-161.

10. T. Aleroev, S. Erokhin, "Parametric Identification of the Fractional Derivative Order in the Bagley-Torvik Model," *Mathematical Models and Computer Simulations*, no. 11, pp. 219-225.

11. E.R. Keharsaeva, V.G. Pirozhkov, "Modeling of changes in the deformation-strength characteristics of asphalt concrete under loading using fractional calculus," *Sat. Proceedings* "*Mechanics of composite materials and structures, complex and heterogeneous media*". Moscow, IPRIM RAS, 2016, pp.104-109.

12. N.N. Staroverov, "Modeling of viscoelastic properties and hysteresis damping of springs made of composite materials," *News of higher educational institutions. Engineering*," 2011, no. 8, pp. 25-34.

13. A. Elsayed, "The using of fractional calculus to study the deformation-strength characteristics of polymer concrete: Candidate of Physics and Mathematics," *MIPT*, Moscow 2023, 153 p.

14. T. Aleroev, "On completeness of the system of eigenfunctions for one differential operator of fractional order," Differential Equations, no. **36**:6. 2000.

15. A.M. Nakhushev, "Fractional Calculus and Its Application," Moscow: Fizmatlit, 2003, p. 272.

К ПРОБЛЕМЕ ОПТИМИЗАЦИИ СВОЙСТВ ВЯЗКОУПРУГИХ МАТЕРИАЛОВ С ПОМОЩЬЮ УРАВНЕНИЯ БЕГЛИ-ТОРВИКА

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Аннотация
В последнее время дробное исчисление находится в центре внимания многих исследователей в области науки и техники, так как более детальное исследование физических процессов приводят к необходимости усложнения математических моделей, их описывающих, и, следовательно, к исследованию поведения решений дифференциальных уравнений, содержащих, наравне с "обычной", или "классической", производной, еще и дробную. К процессам такого рода могут быть отнесены: исследования сплошных сред с памятью, фильтрация жидкости в средах с фрактальной геометрией, физические аспекты стохастического переноса и диффузии, математические модели вязкоупругого тела, модели затухающих колебаний с дробным демпфированием (например, колебания горных пород при землетрясениях или колебания наноразмерных сенсоров), модели нелокальных физических процессов и явлений фрактальной природы; климатические модели и т.д. В работе изучаются краевые задачи для уравнения движения осциллятора с вязкоупругим демпфированием (уравнение Бегли-Торвика) в случае, когда порядок демпфирования больше нуля, но меньше двойки. Такие задачи моделируют многие физические процессы, в частности, колебание струны в вязкой среде, изменение деформационно-прочностных характеристик полимербетона при нагружении и др. Данной работа посвящена оптимизации параметрического управления модели Бегли-Торвика. Предложен принципиально новый, эффективный алгоритм позволяющий оценить параметры модели реальных материалов.

Ключевые слова: *уравнение Бегли-Торвика, дробное исчисление, вязкоупругость, краевая задача, эластомерный подшипник, идентификация параметра, вязкоупругая среда.*

Литература

1. *Георгиевский Д.В., Шамолин М.В.* Заседания семинара механико-математического факультета МГУ им. М. В. Ломоносова "Актуальные проблемы геометрии и механики" им. проф. В. В. Трофимова под руководством С. А. Агафонова, Д. В. Георгиевского и М. В. Шамолина" // Геометрия и механика, Итоги науки и техн. Сер. Соврем. мат. и ее прил. Темат. обз., 205, ВИНИТИ РАН, М., 2022. С. 3-9.
2. *Кох К.Г., Келли Дж.М.* Применение дробных производных к сейсмическому анализу моделей с изолированным основанием. 1990 https://doi.org/10.1002/eqe.4290190207
3. *Бегли Р., Торвик П.* Дробное исчисление - иной подход к анализу вязкоупругих демпфирующих конструкций // Aiaa Journal, № 21. С. 741-74, 1983.
4. *Кириллов А.М., Завьялов М.А.* Моделирование процессов энергообмена в системе дорожное покрытие – транспортное средство // Инженерно-строительный журнал, №5, 2015.
5. *Яшина М.В., Поспелов П.И., Кутейников И.А.* Влияние распределения транспортного потока на колейность дорожных покрытий. Труды XIV Всероссийской мультиконференции по проблемам управления МКПУ-2021 // Материалы XIV мультиконференции в 4 томах. Том 4. Ростов-на-Дону – Таганрог, 2021. С. 107-110.
6. *Абышов Р.Г., Буслаев А.П., Куприянов Ю.Д., Яшина М.В.* Распределенная система мониторинга содержания автомобильных дорог // Вестник МАДИ, № 1(24), 2011. С. 79-85.
7. *Буслаев А.П., Замотайлов О.В., Поспелов П.И., Яшина М.В.* О восстановлении слоев дорожной одежды по данным подповерхностной радиолокации (ч.1) // Вестник МАДИ(ГТУ), 2009. С. 89-96.
8. *Алероев Т.С., Алероева Х.Т.* Задачи типа Штурма-Лиувилля для дифференциальных уравнений с дробными производными. Т.2 Дробные дифференциальные уравнения; под ред. Кочубея А. и Лучко Ю.; Изд. De Gruyter: Германия; США, 2019. С. 21-46.
9. *Огородников Е.Н., Радченко В.П., Унгарова Л.Г.* Математические модели нелинейной вязкоупругости с операторами дробного интегро-дифференцирования // Вестник Пермского национального исследовательского политехнического университета. Механика, 2018, № 2. С. 147-161.
10. *Алероев Т.С., Ерохин С.В.* Параметрическая идентификация порядка дробной производной в модели Бегли-Торвика // Матем. моделирование, № 30:76 2018. С. 93-102.
11. *Кехарсаева Э.Р., Пирожков В.Г.* Моделирование изменения деформационно-прочностных характеристик асфальтобетона при нагружении с помощью дробного исчисления // Сб. трудов "Механика композиционных материалов и конструкций, сложных и гетерогенных сред", М., ИПРИМ РАН, 2016. С. 104-109.
12. *Староверов Н.Н.* Моделирование вязкоупругих свойств и гистерезисного демпфирования рессор из композиционных материалов // Известия высших учебных заведений. Машиностроение, 2011, №8. С. 25-34.
13. *Елсайед Асмаа Елсайед Махди.* Применение дробного исчисления для исследования деформационно прочностных характеристик полимербетона: кнд.физ.мат.н-ук.1.2.2 / Мос.гос. физ. тех. у-нт. Москва, 2023. 153 с.
14. *Алероев Т.С.* К проблеме о нулях функции Миттаг-Лефлера и спектре одного дифференциального оператора дробного порядка // Дифференц. уравнения, том 36, №9, 2000ю С. 1278-1279.
15. *Нахушев А.М.* Дробное исчисление и его применение, Издательство Физматлит, 2003. 272 с.

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