THE CALCULATION OF THE SOLUTION OF MULTIDIMENSIONAL INTEGRAL EQUATIONS WITH METHODS MONTE CARLO AND QUASI-MONTE CARLO

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The article considers an approach based on the random cubature method for solving both single and multidimensional singular integral equations, Volterra and Fredholm equations of the 1st kind, for ill-posed problems in the theory of integral equations, etc. A variant of the quasi-Monte Carlo method is studied. The integral in an integral equation is approximated using the traditional Monte Carlo method for calculating integrals. Multidimensional interpolation is applied on an arbitrary set of points. Examples of applying the method to a one-dimensional integral equation with a smooth kernel using both random and low-dispersed pseudo-random nodes are considered. A multidimensional linear integral equation with a polynomial kernel and a multidimensional nonlinear problem – the Hammerstein integral equation – are solved using the Newton method. The existence of several solutions is shown. Multidimensional integral equations of the first kind and their solution using regularization are considered. The Monte Carlo and quasi-Monte Carlo methods have not been used to solve such problems in the studied literature. The Lavrentiev regularization method was used, as well as random and pseudo-random nodes obtained using the Halton sequence. The problem of eigenvalues is solved. It is established that one of the best methods considered is the Leverrier-Faddeev method. The results of solving the problem for a different number of quadrature nodes are presented in the table. An approach based on parametric regularization of the core, an interpolation-projection method, and averaged adaptive densities are studied. The considered methods can be successfully applied in solving spatial boundary value problems for areas of complex shape. These approaches allow us to expand the range of problems in the theory of integral equations solved by Monte Carlo and quasi-Monte Carlo methods, since there are no restrictions on the value of the norm of the integral operator. A series of examples demonstrating the effectiveness of the method under study is considered.

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1. Method description

Consider the Fredholm integral equation (IE),

$$\mu u(x) - \lambda \int_V K(x,y)u(y)dy = f(x),\ x \in V, \qquad (1)$$

where $u(x)$ is the desired function, the points of the domain from the V $m-$dimensional Euclidean space, and $\mu, \lambda -$ some real or complex numbers, $K(x,y) -$ the kernel of the integral operator, the $f(x) -$ free term.

Suppose that we know the random n points of the domain $V: y^1 = (y_1^1,...,y_m^1),...,\ y^n = (y_1^n,...,y_m^n)$, obtained from a distribution with densit $p(y), y \in V$.

The normalization condition: $\int_V p(y)dy = 1$. The integral in (1) can be approximated by the traditional scheme for calculating integrals by the Monte Carlo method [1]:

$$\int_V K(x,y)u(y)dy \approx \frac{1}{n}\sum_{j=1}^{n} S_j(x),\ x \in V,$$

where $S_j(x) = \dfrac{K(x,y^j)u(y^j)}{p(y^j)}$.

Let us rewrite (1) in the equivalent form:

$$\mu u(x) - \frac{\lambda}{n}\sum_{i=1}^{n} S_i(x) - \lambda R_n(x) = f(x),\ x \in V, \qquad (2)$$

where $R_n(x)$ – is the residual term of the Monte Carlo integration formula:

$$\int_V K(x,y)u(y)dy = \frac{1}{n}\sum_{j=1}^{n} S_j(x) + R_n(x).$$

We use points $y^1 = (y_1^1,...,y_m^1)$,..., $y^n = (y_1^n,...,y_m^n)$, as nodes of collocations in the well-known computational method, which gives from (2) the corresponding SLAE for approximate solutions of the points in question:

$$\mu u_i - \frac{\lambda}{n}\sum_{j=1}^{n} \frac{K(y^i,y^j)}{p(y^j)} u_j = f_i,\ u_i \approx u(y^i),\ i = 1,...,n. \qquad (3)$$

Since the residual term of the quadrature sum of Monte Carlo method with any given probability tends to zero as the number of knots tends to infinity, it is reasonable to assume that under a sufficiently smooth kernel and bounded operator inverse to the operator of integral equation (1), the SLAE solution (3) converges to exact in one of probability measures. In the literature the corresponding convergence issues are dealt with for Neumann series summation problem [1] and for so called semi-statistical method [2,4,5].

2. Examples of applications of the method

2.1. A one-dimensional IE with a smooth kernel

Input data for the model problem: $f(x) = 1$ $K(x,y) = sin(x) - sin(y)$; the integration interval $[0,\pi]$. The exact solution is $u(x) = (\pi sin(x) - 1) / \left(\dfrac{\pi^2}{2} - 3\right)$; $\lambda = 1$, $\mu = 1$. The solution is found using Monte Carlo methods for the usual uniformly distributed pseudo-random grid and the corresponding modified low-dispersion sequence, quasi Monte Carlo (the nodes of the grid are points olow-dispersion Halton $H_s(i)$ sequence, $i = 1,...,n$; $s -$ prime number), and the quadrature central rectangular method.

The results are shown in Figers 1 and 2, where the solid line corresponds to the exact solution.

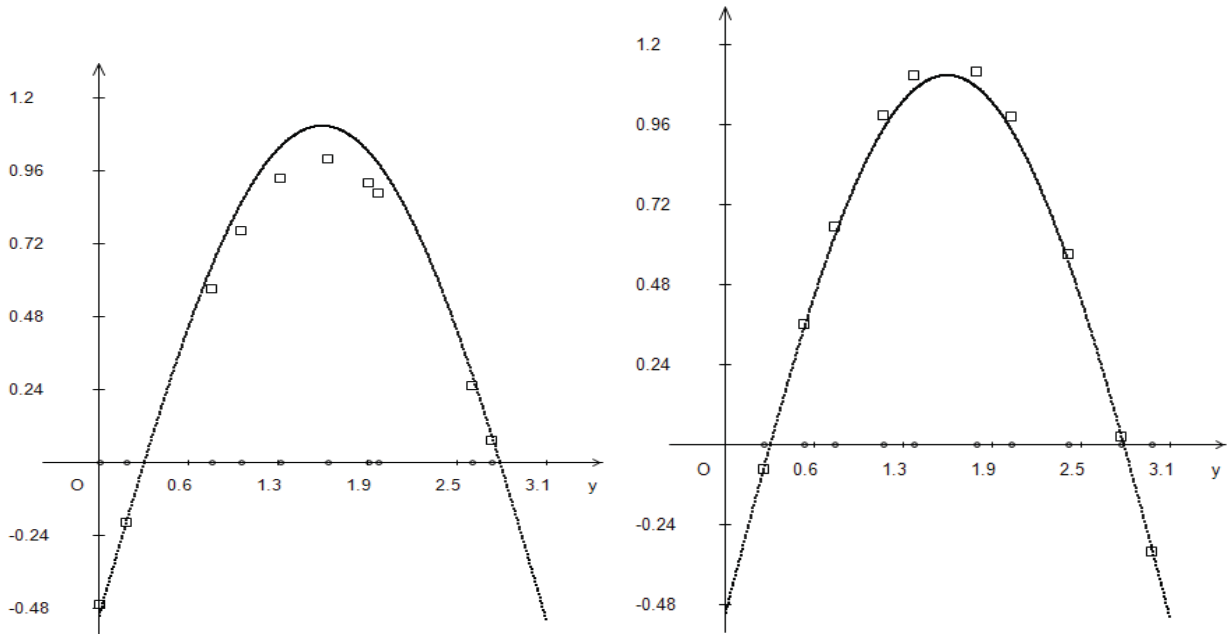

Figure 1. Solution of IE (1) by Monte Carlo method. Left: *n=10*, pseudorandom nodes; right: *n=10*, modified pseudorandom sequence

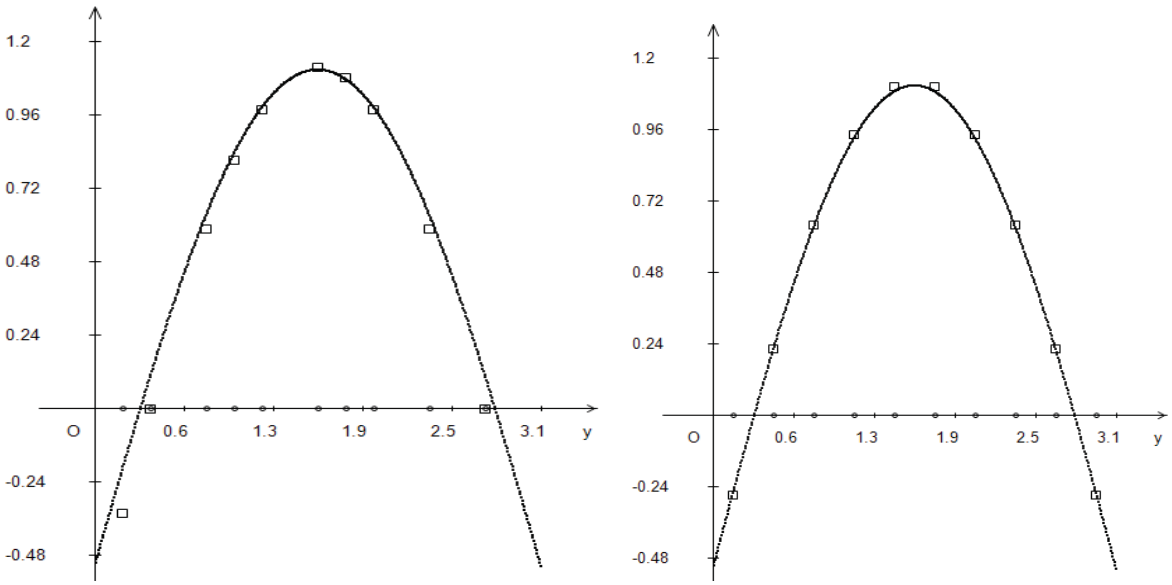

Figure 2. Numerical solution of IE (1). Left: *n=10*, Halton $H_i(2)$ sequence; right: *n* = 10, the central rectangles formula

2.2. Multivariate linear IE with polynomial kernel

Initial data f the model roblem: $K(x,y)=x_1\dots x_m y_1\dots y_m$, $f(x)=x_1\dots x_m+g(x_1\dots x_m)^2$; $g=10$; The area of integration is *an m-dimensional* cube $D^m=\{0<x_1,\dots,x_m,y_1,\dots,y_m<1\}$.

Exact solution $u(x)=c_0x_1\dots x_m+gx_1\dots x_m(x_1\dots x_m+c_1)$; $c_0=0.5$, $c_1=(3/4)^m$, $\lambda=-3^m$, $\mu=1$. Results of three consecutive calculations of the solution for the size of the area and *m* = 10 the number of nodes *n* = 10: exact solution (0.02375, 0.01527, 0.00363, 0.04009, 0.04210, 0.08175, 0.00694, 0.03348, 0.03155, 0.01348); approximate solution (0.02679, 0.01758, 0.00448, 0.04425, 0.04638, 0.08800, 0.00831, 0.03722, 0.03516, 0.01561).

The solution error of the normal solution is about 11% l_1. On the next two calculations of the solution the error is much smaller: about 1% and 0.4%.

2.3. Multivariate nonlinear problem - Gammerstein IE

Consider a non-linear Fredholm IE of the form:

$$\mu u(x)-\lambda\int_V K(x,y)b(u(y),y)dy=f(x),\ x\in V, \qquad (4)$$

the corresponding quadrature formula:

$$\mu u_i-\frac{\lambda}{n}\sum_{j=1}^{n}\frac{K(y^i,y^j)}{p(y^j)}b(u_j,y^j)=f_i,u_i\approx u(y_i),$$
$$i=1,\dots,n. \qquad (5)$$

As a test problem, consider a FE:

$$\mu u(x)-\lambda\int_V K(x,y)u^2(y)dy=f(x),\ x\in D^m, \qquad (6)$$

$K(x,y)=x_1\dots x_m y_1\dots y_m$, $f(x)=x_1\dots x_m$.

There are two exact solutions $u(x)=c^{(k)}x_1\dots x_m$; $c^{(1,2)}=\frac{1\pm\sqrt{1-\lambda 4^{-m+1}}}{2\cdot 4^{-m}\lambda}$, $u(x)=c^{(k)}x_1\dots x_m$; Monte Carlo solution results for the following initial data: $m=10$, $r=100$; $\lambda=-r3^m$, $r=1\dots100$, $\mu=1$, the initial approximation $u_i^{(0)}=f(y_i)$, $i=1,\dots,n$ is . The set of coordinates of nodes in the random grid was simulated for density of distribution: $p(x)=4^m(x_1\dots x_m)^3$, $x\in D^m$, so the coordinate values were found using the appropriate formula of the inverse function method: $\gamma_k^i=(\gamma_k^i)^{1/4}$, γ_k^i – random number for the corresponding node and its coordinates. The coordinate values are shown in Table 1. The results of Newton iterations are given in Tables 2 and 3.

Table 1

Coordinates of pseudo-random grid nodes

Node number	Node coordinates							
1	0.92	0.8	0.73	0	0.78	0.94	.80	0.63
2	0.89	0.8	0.72	0	0.65	0.71	.81	0.86
3	0.73	0.7	0.64	0	0.83	0.75	.91	0.73
4	0.95	0.9	0.98	0	0.53	0.90	.75	0.91
5	1.00	0.8	0.84	0	0.99	0.95	.99	0.78
6	0.67	0.6	0.87	0	0.45	0.80	.88	0.75
7	0.59	0.8	0.60	0	0.85	0.97	.28	0.47
8	0.48	0.8	0.93	0	0.74	0.94	.79	0.93
9	0.72	0.9	0.76	0	0.98	0.88	.81	0.69
10	0.73	0.8	0.33	0	0.95	0.87	.83	0.71

Table 2

Results for the first iteration

Node number	Errors of iterations	Approximate solution	A precise solution
1	-0.0403	0.04755	0.03006
2	-0.0305	0.03600	0.02276
3	-0.0322	0.03796	0.02400
4	-0.0610	0.07193	0.04547
5	-0.0673	0.07928	0.05012
6	-0.0131	0.01543	0.00975
7	-0.0040	0.00481	0.00304
8	-0.0654	0.07712	0.04875
9	-0.0377	0.04443	0.02809
10	-0.016	0.01919	0.01213

After the first iteration, the solution error was 58% for the - non-arithmetic means. After the fourth iteration, the error was less than 0.01%.

Table 3

Results for the fourth iteration

Node number	Errors of iterations	Approximate solution	A precise solution
1	-0.0000	0.03006	0.03006
2	-0.0000	0.02276	0.02276
3	-0.0000	0.02400	0.02400
4	-0.0001	0.04547	0.04547
5	-0.0001	0.05012	0.05012
6	-0.0000	0.00975	0.00975
7	-0.0000	0.00304	0.00304
8	-0.0001	0.04875	0.04875
9	-0.0000	0.02809	0.02809
10	-0.000	0.01213	0.01213

An appropriate choice of initial approximation (e.g, $u_i^{(0)} = -50 f(y_i),\ i = 1,\ldots,n$) makes it possible to find a second solution of the given IE. The results of the calculations are similar to those considered above and therefore are not given.

2.4. Multidimensional FEs of the first kind and their solution using regularisation

Monte Carlo and quasi-Monte Carlo methods for solving similar problems have not been described in the reviewed literature. Initial data of the model problem: — $D^m = \{0 < x_1,\ldots,x_m, y_1,\ldots,y_m < 1\}$, $m=10$, $K(x,y) = x_1 \ldots x_m y_1 \ldots y_m$.

The problem under consideration belongs to incorrect problem, which is typical for first order Fredholm linear equation. The nonlinear term plays the role of a perturbation. For solving this problem, we used the well-known method of Lavrenteev regularization; in accordance with this method, we converted the IE to a second kind of IE with a small coefficient of the solution function μ = 0.01; the exact solution of the IE takes the form $u(x)=c_0 x_1 \ldots x_m + g(x_1 \ldots x_m)^{2/\mu}$; its simplified solution: $u_0(x)=c_0 x_1 \ldots x_m$; $c_0=(1+\lambda g \cdot 4^{(-m)}/\mu)/(\mu-\lambda \cdot 3^{(-m)})$, $\lambda=(-3)^{m.}$ The set of coordinates of the random grid nodes was modeled for the density of the distribution: $p(x) = c^m (x_1 \ldots x_m)^{c-1},\ x \in D^m$.

The coordinate values were found using the formula of the inverse function method: $\gamma_k^i = (\gamma_k^i)^{1/c}$, the γ_k^i – random number for the corresponding node and its coordinates in the Monte Carlo method or the Halton sequence number $H_i(2)$ in the quasi-Monte Carlo method. The results of calculations at n = 100 10 trials for each method are given in Tables 4 and 5. The parameter "degree c" is a characteristic of the distribution density of the integration nodes.

Table 4

Results for the pseudo random sequence

Degree, C	Simplified solution		Numerical solution	
	Average – sinfulness, %	Standard errors, %	Average sinfulness, %	Standard errors, %
1	13	3	125	47
2	11	10	52	33
3	9	1	6	3
4	9	1	3	3
5	9	0.4	6	4

Table 5

Results for the Halton sequence

Degree, C	Simplified solution		Numerical solution	
	Average – sinfulness, %	Standard errors, %	Average – sinfulness, %	Standard errors, %
1	12	3	166	64
2	10	5	26	16
3	9	2	9	6
4	9	1	3	2
5	9	1	6	4

The data in Tables 4 and 5 shows that the accuracy of both methods is approximately the same. The node density was chosen according to the type of simplified solution obtained if the parameter g is zero. The degree is c = 4optimal, which is explained by the fact that the integrating cubic terms make the main contribution to the integrate function.

2.5. The eigenvalue problem

One of the advantages of this method is the ability to numerically solve the complete eigenvalue problem for a multidimensional integral operator. The approximate eigenvalues and eigenfunctions are found by solving the complete eigenvalue problem for the corresponding matrix of quadrature formula (3). According to the known theory, the limit set oeigenvalues of matrices approximating an integral operator includes the set of eigenvalues of that integral operator.

There are many methods for solving the complete eigenvalue problem for matrices: the Jacobi (rotation), Krylov, Leverier-Faddeev, Danilevsky and some other methods. As a result of the study it was found that one of the best of the considered methods is the Leverier-Faddeev method: the Jacobi method assumes a symmetric matrix, the Krylov method is economical, but its area of applicability is limited (for the considered problems we could not find a solution by this method), the Danilevsky method is relatively complicated.

The results of solving the problem for various numbers of quadrature nodes are presented in the table. They show that in spite of low approximation accuracy of the function - solution of FE, the approximation accuracy of the eigenvalues is satisfactory even for small n.

As a model problem (1) is considered for $f(x)=x+5;\ 0<x<1;\ \lambda=1.5,\ \mu=1\quad K(x,y)=\sin(5x+5y)$. The kernel of the IE is degenerate, and the exact solution is $u(x)=C_1\cos(5x)+C_2\sin(5x)+C_0(x)$, where C_1,C_2 – are some constants, $C_0(x)$ – one of zeros of the integral operator (corresponding to the eigenvalue $z_0=1$). The IE is reduced to a SLAE with respect to the constants C_1,C_2 with matrix:

$$a_{11}=1-\frac{1.5}{20}(1-\cos(10)),a_{12}=-1.5\left(0.5-\frac{\sin(10)}{20}\right),$$

$$a_{21}=1.5\left(0.5+\frac{\sin(10)}{20}\right),a_{22}=1-\frac{1.5}{20}(1-\cos(10))\cdot$$

The coefficients of the characteristic polynomial of this matrix are $det(A-zE)=z^2-p_1z-p_2\quad p_1=1.72414$, $p_2=-0.18233$; eigenvalues (zeros of the polynomial) are $z_1=1.61096$, $z_2=0.11318$; Results obtained by three methods are shown in Table 6 (method 1 – rectangular method, method 2 – onte Carlo method, method 3 – quasi Monte Carlo method).

Table 6

Results of eigenvalue calculations

N	Method 1		Method 2		Method 3	
	z_1	z_2	z_1	z_2	z_1	z_2
5	1.58452	0.08765	1.75172	0.25856	1.50610	0.04589
10	1.60494	0.10736	1.51632	0.04268	1.63468	0.13484
15	1.60833	0.11063	1.78386	0.28386	1.59268	0.09551
20	1.60950	0.11176	1.55148	0.07998	1.56529	0.10672

The best results are obtained with the rectangular method, with a slightly worse accuracy with the quasi Monte Carlo method. The results of the Monte Carlo method are less reliable. This example reveals an interesting property: the convergence for eigenvalues is much better (hundreds of times better) than for the IE solution function.

2.6. Singular IE

2.6.1. Method for highlighting a feature

Consider a Fredholm IE (1) with a singular kernel $K(x,y)$, unbounded at *x* = *y*. For the solution we apply the method of singular point extraction:

$$\mu u(y^i)-\lambda\int_{V_i}K(y^i,y)u(y)dy-\lambda\int_{V\setminus V_i}K(y^i,y)u(y)dy=f(y^i) \tag{7}$$

$i=1,...,n.$ The kernel of the of the integral in (7) is unbounded. The dimensions of the domain are V_i relatively small of the order of $O(n^{-1/m})$, at $n\to\infty$, the variation $u(y)$ in this domain tends to zero, and the integral can be calculated approximately by the formula:

$$\int_{V_i}K(y^i,y)u(y)dy\approx\int_{V_i}K(y^i,y)dyu(y^i)=J_iu(y^i),\ y^i\in V_i\cdot$$

The point y_i it is feasible to find the point in the centre of V_i the region. The kernel of the second integral in (7) is regular, so that the integral can be calculated by the Monte Carlo method:

$$\int_{V\setminus V_i}K(y^i,y)u(y)dy\approx\frac{1}{n-n_i}\sum_{j\in V\setminus V_i}S_j(y^i),$$

where $S_j(y^i)=\frac{K(y^i,y^j)}{p(y^j)}u(y^j)$, n_i – number of points from the sample, $y^1=(y_1^1,\ldots,y_m^1)$, $y^n=(y_1^n,\ldots,y_m^n)$ belonging to the closed region $\overline{V_i}$, the density $p(y)$ is normalized to one in the region $V\setminus V_i$. As a result, we obtain from (7) an appropriate SLAE for finding approximate values of solution in $u_i\approx u(y^i)$ nodes of random grid:

$$(\mu-\lambda J_i)u_i-\frac{\lambda}{n-n_i}\sum_{j\in V\setminus V_i}\frac{K(y^i,y^j)}{p(y^j)}u_j=f_i,\ i=1,\ldots,n. \tag{8}$$

Singular IE with Cauchy kernel

$$\mu u(x)-\lambda\int_0^1(x-y)^{-1}u(y)dy=x,\ x\in(0,1), \tag{9}$$

$$V_i=\{x\in(0,1):|x-y_i|<kh/2\},\ i=1,\ldots,n;\ h=\frac{c}{n},$$

c,k – some positive constants, $\lambda=0.2$ $\mu=1$. To solve (9), we considered: A) Monte Carlo method. Random uniformly distributed on the interval (0,1) nodes (also modified distribution with screening of excessively close points). B) The quasi-Monte Carlo method. The nodes of the grid are points of low-dispersion Halton $H_s(i)$, $i=1,\ldots,n$ consistency; S — a prime number. C) Central rectangular and trapezoidal methods.

Figure 3 shows plots of the numerical solution of IE (9) for pseudo-random distribution of integration points. If pseudo-random points are too close together, the accuracy tends to decrease and the number of conditioning of the SLAE matrix increases. To make the distribution more uniform, the sequence of points has been modified by eliminating excessively close points.

Figure 4 shows graphs of numerical solution of IE (9) by quasi-Monte Carlo method. The points of the Halton sequences [1] were chosen as the grid nodes. The calculation results show that increasing the size of the partitioning element containing the kernel feature in some limits leads to improvement of the solution. However, a further increase in the size of the considered element leads to a decrease in accuracy due to the error of approximation of the solution in this element.

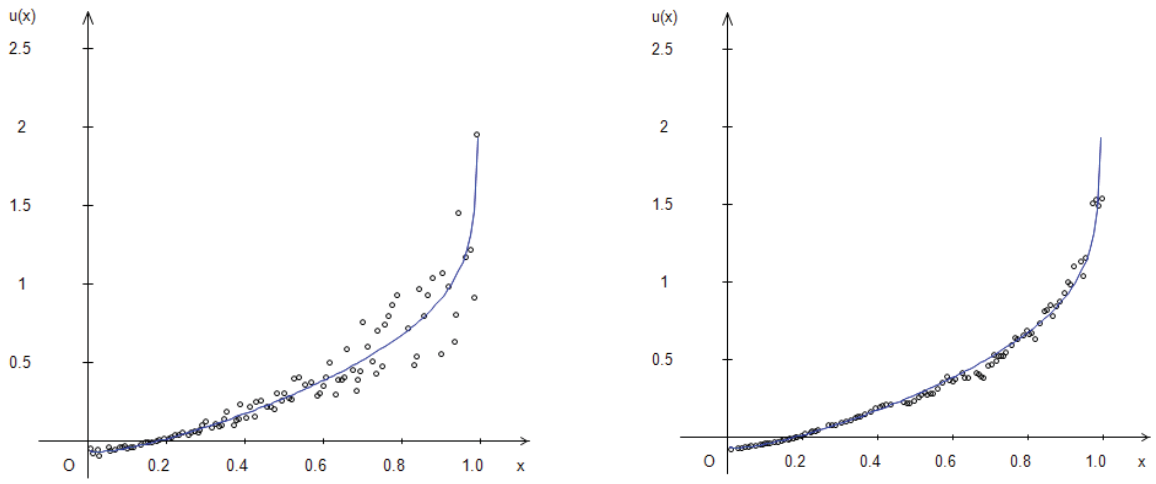

Figure 3. Approximate solution of IE (9). The solid line is the central rectangle method (n = 100), the circles are the Monte Carlo method for 100 pseudo-sluice points (left k = 1; right k = 5)

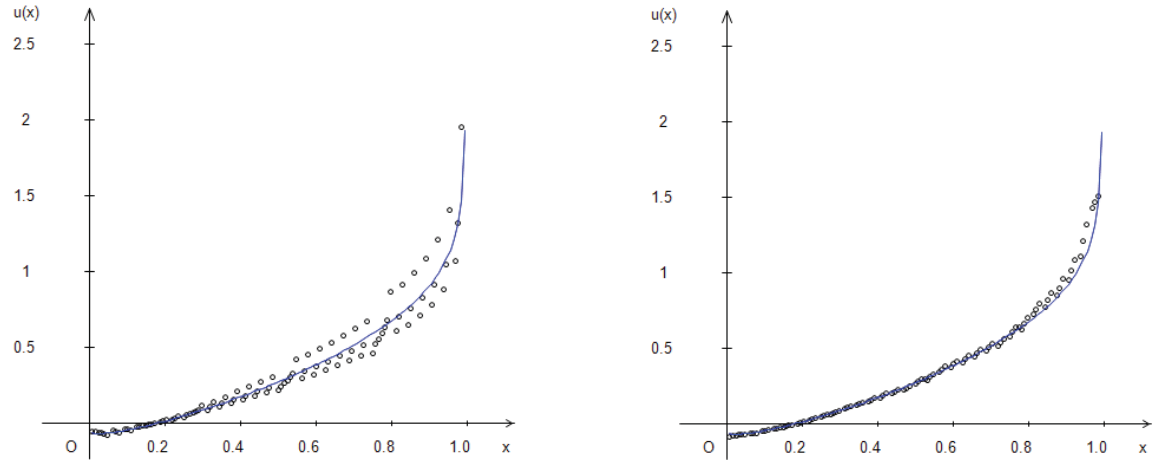

Figure 4. Approximate solution of IE (9). The solid line is the central rectangle method (n = 100), the circles are the quasi-Monte Carlo method for 100 points of the Halton sequence (left k = 1; right k = 5)

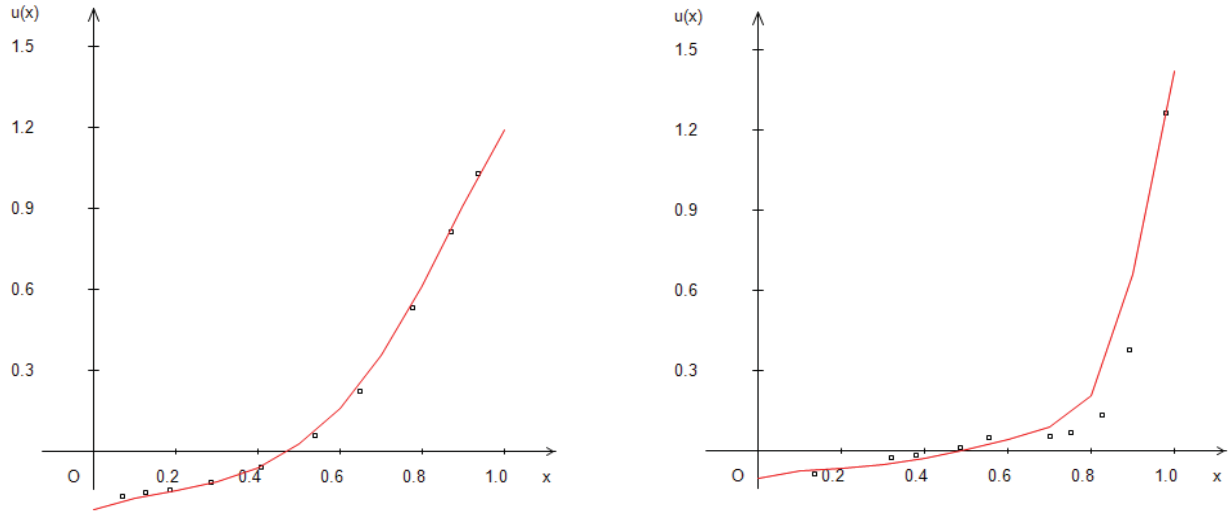

Figure 5. Approximate solution of IE (9). The solid line is the central rectangle (n = 10) method, the circles are the Monte Carlo method for 10 pseudo-random points (left $\varepsilon = \sqrt{0.1}$, right $\varepsilon = 0.1$)

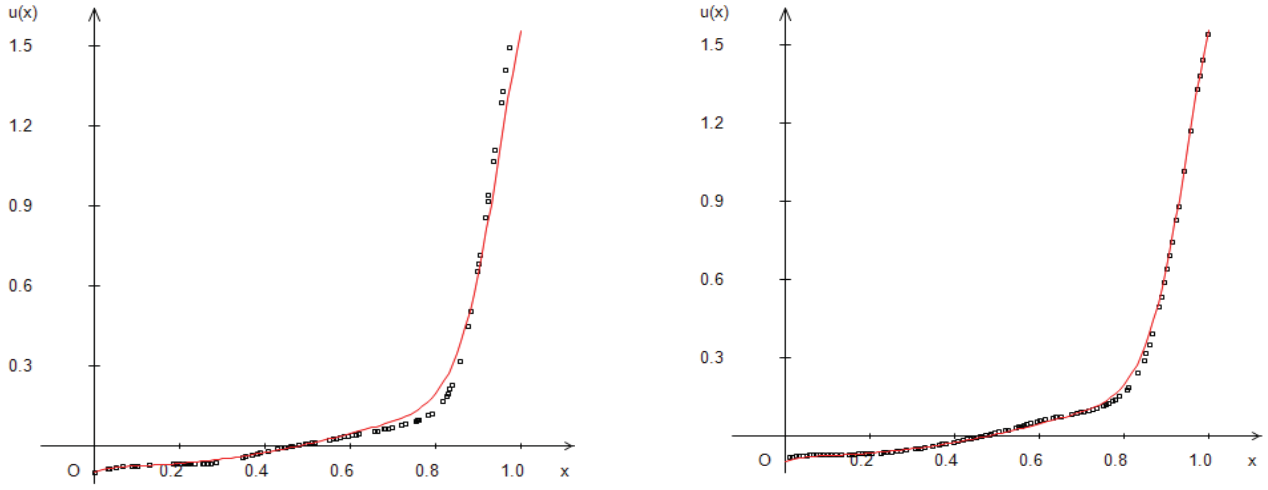

Figure 6. Approximate solution of IE (9) at $\varepsilon = 0.1$. The solid line is the central rectangle method (n=100), the circles are the Monte Carlo method for 100 pseudo-random points (the sequence of points is modified on the right)

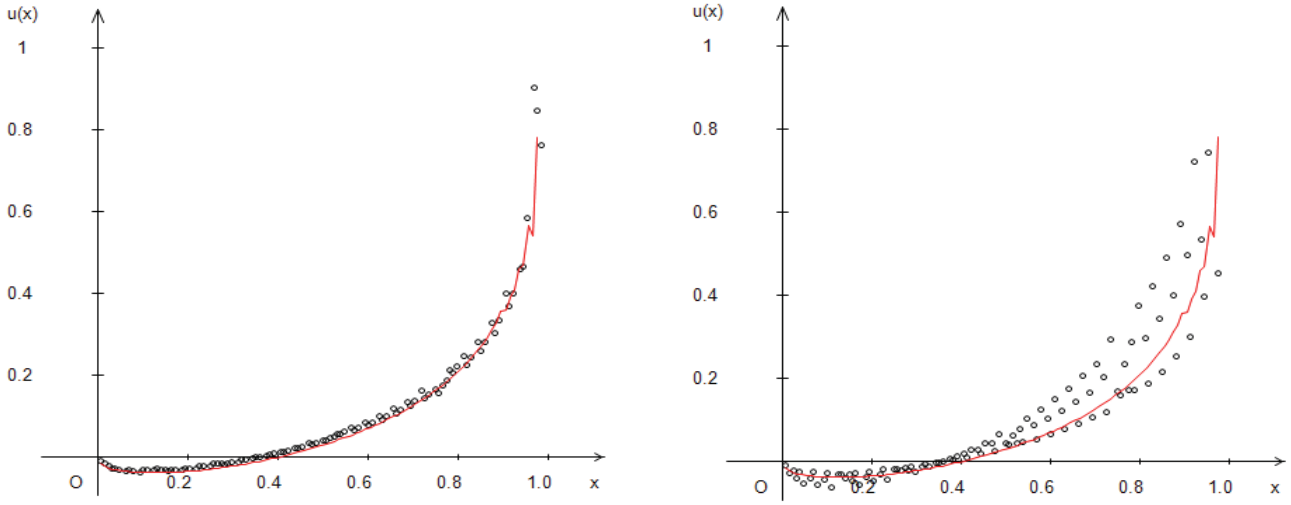

Figure 7. Solution of IE (9) at n=100. The solid line is the central rectangle method (with 2-point smoothing), the circles are the quasi-Monte Carlo method for the Halton sequence (left with smoothing)

2.6.2. Method based on parametric kernel regularisation

In equation (9) the singular kernel is replaced $K(x,y)=(x-y)^{-1}$ by an asymptotically close regular expression $K(x,y)=(x-y)/\left[(x-y)^2+\varepsilon^2\right]\varepsilon\to 0$. Consider the results of the numerical solution of (9) for $\lambda=1, \mu=1$, shown in Figurs. 5, 6.

Regularising the kernel and modifying the pseudo-random uniform sequence by eliminating excessively close points leads to a significant improvement in the result. The disadvantage of the method is its strong dependence on the values of a small parameter ε.

2.6.3. Interpolation-projection method

In this method, the approximate solution is found as a sum of

$$u(x)\approx\sum_{j=1}^{n}C_j\varphi(x-y_j), \tag{10}$$

where $\varphi(x)$ – the basis function, for example, is a widely used finite piecewise linear function $C_j -, \varphi(x)=max(0,1-2|x|/h)$, $h\approx\frac{1}{n}$, the unknown coefficients. Taking into account (10) from (1) follows from

$$\mu\sum_{j=1}^{n}C_j\mu(x-y_j)\approx\lambda\sum_{j=1}^{n}C_j\int_0^1K(x,y)\mu(y-y_j)dy+f(x),\ 0<x<1. \tag{11}$$

Using (11) we find an SLAE for the solution of the unknown coefficients of expression (10) by interpolation method

$$\mu\sum_{j=1}^{n}C_j\varphi\left(y_i-y_j\right)=\lambda\sum_{j=1}^{n}C_j\int_0^1 K\left(y_i,y\right)\varphi\left(y-y_j\right)dy+f\left(y_i\right),$$
$$i=1,\ldots,n. \qquad (12)$$

The constructed method can still be insufficiently accurate for the solution of IE with singular kernel even at a large number of integration nodes (see Fig. 7, right).

The refinement can be achieved by different methods, for example, based on relation (11) using Galerkin method. However, a more economic approach is to apply smoothing, for example, according to scheme

$$\langle u_i\rangle=\frac{1}{2l+1}\sum_{j=-l}^{l}L_n\left(y_i+\frac{jh}{l};u_1,\ldots,u_n\right). \qquad (13)$$

The relevant effect is illustrated in figure 7 at $l=10$.

Conclusion

This paper investigates a statistical method for solving high-dimensional Fredholm integral equations with both smooth and singulakernels. The approaches under consideration extend the problems of the theory of integral equations solvable by Monte Carlo and quasi Monte Carlo methods because there are no restrictions on the value of the norm of the integral operator. A series of examples demonstrating efficiency of the method under study are considered.

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ПРИБЛИЖЕННОЕ РЕШЕНИЕ МНОГОМЕРНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ МЕТОДАМИ МОНТЕ КАРЛО И КВАЗИ-МОНТЕ КАРЛО

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Аннотация

В статье рассматривается подход на основе метода случайных кубатур для решения как одно, так и многомерных сингулярных интегральных уравнений, уравнений Вольтерра и Фредгольма 1 рода, для некорректных задач теории интегральных уравнений и т.д. Изучен вариант квази-Монте Карло для рассматриваемого метода. Интеграл в интегральном уравнении приближенно вычисляется при помощи традиционной схемы вычисления интегралов методом Монте-Карло. Применяется многомерная интерполяция на произвольном множестве точек. Рассмотрены примеры применения метода к одномерному интегральному уравнению с гладким ядром с использованием как случайных, так и низкодисперсных псевдослучайных узлов. Решено с помощью метода Ньютона многомерное линейное интегральное уравнение с полиномиальным ядром, многомерная нелинейная задача – интегральное уравнение Гаммерштейна. Показано существование нескольких решений. Рассмотрены многомерные интегральные уравнения первого рода и их решение с использованием регуляризации. Решение методами Монте-Карло и квази Монте-Карло подобных задач в изученной литературе не проводилось. Был использован метод регуляризации Лаврентьева, а также случайные и псевдослучайные узлы, полученные при помощи последовательности Хальтона. Решена проблема собственных значений. Установлено, что одним из лучших из рассмотренных методов является метод Леверье-Фаддеева. Результаты решения задачи для различного числа квадратурных узлов представлены в таблице. Исследован подход на основе параметрической регуляризации ядра, интерполяционно-проекционный метод, усредненные адаптивные плотности. Рассмотренные методы могут успешно применяться при решении пространственных краевых задач для областей сложной формы. Рассматриваемые подходы позволяют расширить круг задач теории интегральных уравнений, решаемых методами Монте-Карло и квази Монте-Карло, поскольку отсутствуют ограничения на величину нормы интегрального оператора. Рассмотрена серия примеров, демонстрирующих степень эффективности исследуемого метода.

Ключевые слова: *интегральные уравнения, высокая размерность, метод Монте-Карло.*

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